# Ambiguous Business Cycles: A Quantitative Assessment

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**Abstract**

In this paper, we examine the cyclical dynamics of a business cycle model with ambiguity averse consumers and investment irreversibility using the smooth ambiguity model of Klibanoff *et al.* (2005, 2009). The model differentiates between the sources of ambiguity and ambiguity aversion, and allows for learning about an unobserved cyclical component of TFP. We establish two key results. First, the intertemporal substitution in consumption and leisure operating through the transmission channels of the standard Real Business Cycle model dominates the impact of uncertainty aversion when agents can choose to optimally smooth consumption through investment and hours worked choices in response to labor-augmenting technology shocks. Nevertheless, ambiguity and ambiguity aversion affect endogenous choices through information and learning effects. We also examine sectoral TFP data to further understand the information channel through which ambiguity affects quantities and prices. Based on the behavior of the risk-free rate, the greater the distortions induced by ambiguity and ambiguity aversion, the lower the risk-free rate.

**Keywords:** Ambiguity, ambiguity aversion, information and learning, investment irreversibility, Real Business Cycles.

**JEL Codes: C6, D8, E2**

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# 1 Introduction

What role does ambiguity aversion play in generating cyclical fluctuations and asset price movements? In this paper, we adopt the smooth ambiguity preferences of Klibanoff *et al.* (2005, 2009) to examine the cyclical dynamics of a Real Business Cycle model with irreversible investment and labor augmenting technology shocks. In this framework, the shock to aggregate TFP evolves as a function of a latent variable governing its persistence. Ambiguity is introduced in this framework by assuming that agents are unsure about the distribution of the latent variable in that they cannot distinguish a process that has moderate persistence but high volatility, and one which is less volatile but highly persistent. Ambiguity aversion in agents' preferences then endogenously generates "doubt and pessimism" about the external environment: the consumption and investment decisions of an ambiguity averse agent may be viewed in terms of the decisions of an expected utility maximizing agent with beliefs that are more uncertain and pessimistic relative to those that are based on inference from actual data.

There are now a number of studies that seek to the understand the role of uncertainty in driving the business cycle. Studies such as Bloom *et al.* (2012) or Gilchrist *et al.* (2014) construct alternative measures of uncertainty through the aggregation of individually observed quantities.[1] Yet Ludwigson *et al.* (2016) find that higher uncertainty about real variables in a recession is an endogenous response to other variables, while financial market uncertainty is a likely source of the fluctuations. The more recent literature seeks to identify the role of uncertainty/ambiguity aversion that is generated endogenously through agents' optimal choices. Cagetti *et al.* (2002) study the robustness of agents' decision problems for prices and quantities in a one-sector optimal growth model, where the growth rates of technology are altered by infrequent large shocks and continuous small ones. The paper models robustness by assuming that the decision-maker treats the specification of the technology shock process as an approximation, and acts as if a malevolent player threatens to perturb the actual data generating process relative to the approximating model. They find that the robust motive for precaution-

[1] Specifically, Bloom *et al.* (2012) model uncertainty as the volatility of establishment-level measures of total factor productivity in US 4-digit industries while Gilchrist *et al.* (2014) construct an observed measure of uncertainty as time-varying idiosyncratic volatility in firms' stock returns.

ary savings increases the capital stock, which is also a feature of the smooth ambiguity model that we consider. They also find that factor "risk prices" contain an uncertainty premia under robust decision-making, which are largest when the investor is most unsure about the hidden state, namely, the mean growth rate of technology.

Collard *et al.* (2016) study the historical equity premium using the smooth ambiguity model in an endowment economy framework. They find that ambiguity aversion accentuates the conditional uncertainty embodied in U.S. macroeconomic growth outcomes endogenously. After calibrating the ambiguity aversion parameter to match the risk-free rate, they are able to match the first and second conditional moments of observed return dynamics. Jahan-Parvar and Liu (2012) and Liu and Zhang (2015) also seek to account for asset pricing phenomena under the generalized recursive smooth ambiguity first proposed by Ju and Miao (2012) augmented with adjustment costs in investment and a Markov switching process for aggregate productivity growth.[2] Gallant *et al.* (2015) use macroeconomic and financial data with Bayesian methods to measure the size of ambiguity aversion in a Lucas-type consumption-based asset pricing model.

Another strand of the literature uses the multiple priors utility framework of Gilboa and Schmeidler (1989) to account for cyclical and asset pricing phenomena; see Ilut and Schneider (2014), Bianchi *et al.* (2016), or Nimark (2014). Ilut and Schneider (2012) assume a decision problem in which agents act as if they evaluate plans using a worst case probability drawn from a set of multiple beliefs. They model a loss in confidence as an increase in the set of beliefs from which the worst case beliefs are drawn. The paper incorporates ambiguity and changes in confidence into a New Keynesian business cycle model by assuming that "agents' set of beliefs, such as an innovation to productivity, is parameterized by an interval of means centered around zero." An increase in the width of the interval is associated with a loss of confidence, especially when the worst case mean becomes worse. One problem with this approach is that ambiguity and ambiguity aversion are both determined by the size of the set of possible beliefs.[3] Nimark

[2]For example, Liu and Zhang (2015) seek to establish the cross-correlations between equity returns (or the "variance risk premium") and variables typically affected/indicated by the business cycle in a production economy with ambiguity.

[3]Unlike the smooth model of ambiguity, their approach allows for log-linear solution methods which capture the impact of ambiguity and ambiguity aversion in a first-order manner. These authors argue that a change

(2014) develops a business cycle model which combines higher order beliefs - expectations about the expectations of others - with the existence of a public signal that is more likely to be observed after unusual events. Such 'man-bites-dog' type signals increase uncertainty as well as disagreement among agents, and are able to account for periods of large changes in aggregate activity without large changes in underlying fundamentals.

In this paper, we examine the ability of a calibrated/empirically founded productivity shock process that allows for asymmetric effects in booms versus recessions can generate the cyclical dynamics of real variables such as consumption, investment and output) of the strength and duration observed in the data when coupled with ambiguity and investment irreversibility. In our framework, ambiguity arises from the fact that agents cannot distinguish between two possible processes driving an unobserved temporary component to the aggregate TFP process. Backus *et al.* (2014) have argued that learning together with the role of ambiguity may prove useful for generating the observed business cycle dynamics in that learning provides another source of dynamics, which have been exploited by Collard *et al.* (2012) and Collin-Dufresne *et al.* (2013) to generate significant effects on asset prices. This arises in our framework as agents do not know whether variation in aggregate TFP is driven by a process with high persistence and low volatility, or one with lower persistence but higher volatility, and they must make inferences about the probability of one of these processes being the true process at the same as they infer the behavior of the unobserved temporary component using a Kalman filtering algorithm. In contrast to a situation with learning about an unobserved temporary component drawn from a known probability distribution, ambiguity arises from the problem of inferring the true probability distribution generating the temporary fluctuations in future fundamentals.

Our results reveal two key findings regarding the role of ambiguity aversion with learning in an otherwise standard real business cycle model. First, as Tallarini (2000) or Backus *et al.* (2014) have emphasized in their analysis of cyclical fluctuations generated by recursive non-expected utility or ambiguity-averse preferences, fluctuations in aggregate quantities arise

in confidence behaves like a news shock considered, for example, by Beadry and Portier (2006), Jaimovich and Rebelo (2009), and others. However, the difference between changes in confidence and news shocks is that the latter are followed, at least on average, by a shock realization that corroborates the news, but this need not be the case for changes in confidence.

primarily from changes in intertemporal substitution motives while ambiguity aversion plays a smaller role.[4] Both our quantitative findings together with a log-linear approximation to the social planner's under the assumption of known persistence for the TFP process corroborate this result. Specifically, we find that the intertemporal substitution in consumption and leisure operating through the transmission channels of the standard Real Business Cycle model dominates the impact of uncertainty aversion when agents can choose to optimally smooth consumption through investment and hours worked choices in response to labor-augmenting technology shocks. Second, we show information and learning effects are key to how ambiguity and ambiguity aversion affect endogenous choices. Based on Bayesian estimates of the underlying TFP process, we show that more precise beliefs regarding the unknown process generating the TFP process are associated with lower cyclical variability and greater persistence in the quantity variables. This is a transmission channel that is typically missing in business cycle models.

The informativeness of the underlying TFP process also has implications for the role of ambiguity and ambiguity aversion on the behavior of asset prices and in particular, the risk-free rate. As is well known, an ambiguity averse agent has "as if" beliefs which are more pessimistic relative to a Bayesian learner placed in a similar environment. We show that the greater the endogenous distortions induced by ambiguity aversion, the lower is the risk-free rate in equilibrium. Consequently, one may view the magnitude of the risk-free implied by the model as an endogenous measure of ambiguity. Using data on aggregate TFP as well as estimates of sectoral TFP growth from Bloom *et al.* (2012), we demonstrate that the existence of such distortions depends on the properties of the estimated TFP processes as well as the presence of ambiguity aversion. Interestingly, we find that standard uncertainty measures do not always correlate with ambiguity measures based on similar data.

The remainder of this paper is organized as follows. Section 2 describes the model and the social planner's problem used to generate the optimal quantities. It also describes the smooth

[4]Strzalecki (2013) shows more generally that there is interdependence between ambiguity and the timing of the resolution of uncertainty in models of ambiguity aversion, and that a quantitative assessment is required to disentangle the importance of two effects, which may depend on the calibrated parameters in applications such as ours.

ambiguity preferences and the evolution of beliefs under ambiguous information about exogenous shocks. Section 3 describes the Bayesian estimation of the exogenous processes describing TFP growth. It also describes the simulation results for the model under alternative preference and informational assumptions. Section 4 concludes. Details of the numerical solution method, the Bayesian estimation method, and the log-linearization to the social planner's problem are provided in the Online Appendix .

# 2 The Model

## 2.1 Agent's preferences: recursive smooth ambiguity

We begin by setting out a dynamic, recursive version of the smooth ambiguity averse preferences as developed by Klibanoff *et al.* (2005, 2009). This model is based on the state space, which is the set of all observation paths emanating from an initial state $s_0$. Thus, the state at date $t$ is denoted $s^t = (s_0, s_1, \cdots, s_t)$, where $s_t \in \Upsilon_t$. Agents choose consumption/investment plans $f$, each of which associates a payoff to the node $s^t$ in the event tree. The agent is uncertain about the stochastic process governing the probabilities on the event tree. This uncertainty is indexed by the parameter $\theta \in \Theta$, which denotes the set of unobservable parameters. The probability that the next observation will be $s_{t+1}$, given that the node $s^t$ has been reached on the event tree, is given by $\pi_\theta(s_{t+1}|s^t)$. The agent further has a prior $\mu(\theta)$ for $\theta \in \Theta$. Using the representation in Klibanoff *et al.* (2005, 2009), the recursive smooth ambiguity preferences over plans $f$ at the node $s^t$ are updated and represented as

$$V_{s^t}(f) = u(f(s^t)) + \beta\phi^{-1}\left[\int_\Theta \phi\left(\int_{\Upsilon_{t+1}} V_{(s^t,s_{t+1})}(f)d\pi_\theta(s_{t+1}|s^t)\right)d\mu(\theta|s^t)\right], \tag{2.1}$$

where $V_{s^t}(f)$ is a recursively defined direct value function, $u(\cdot)$ characterizes attitudes towards risk, $\beta$ is a discount factor, $\phi(\cdot)$ is a function characterizing the agent's ambiguity attitude, and $\mu(\cdot|s^t)$ denotes the Bayesian posterior. A concave $\phi$ characterizes ambiguity aversion, which is defined to be an aversion to mean-preserving spreads in the distributions of expected utility values. The model also does not, in general, allow a reduction between the second-order beliefs $\mu$ and the first-order probabilities denotes by $\pi_\theta$; such a reduction occurs only in the case of a linear $\phi$ which represents an ambiguity neutral Bayesian expected utility maximizer.

## 2.2 Production and capital accumulation

To begin we assume a one-sector economy where the production function of the firm is given by

$$y_t = k_t^a (A_t n_t)^{1-a}, \quad 0 < a < 1, \tag{2.2}$$

where $A_t$ is a technology shock. The firm's capital stock evolves as

$$k_{t+1} = (1-\delta)k_t + i_t, \quad 0 < \delta < 1. \tag{2.3}$$

Finally, we assume that investment is irreversible, $i_t \geq 0$. As we show in a decentralized version of the model, this assumption amounts to assuming that there is no market for used capital or, equivalently, its price is zero. While it may be argued that investment irreversibility does not hold at the aggregate level, we choose to use this investment friction in place of the standard adjustment costs model, as the presence of irreversible investment will lead to an endogenous cost of adjustment. (See Demers *et al.* (2003).)

Uncertainty in this economy is assumed to driven by the stochastic behavior of productivity growth. As in Collard *et al.* (2016), we assume there is a long run average growth rate of productivity, $\bar{g}$, and a deviation from it, $x_{k,t+1}$, which is assumed to follow a persistent stochastic process. This specification of the technology process is similar to the models of long-run risk proposed by Bansal and Yaron (2004) and Croce (2010). However, the business cycle effect on productivity, $x_{k,t+1}$, is not observed directly, and agents are unsure the value of the persistence parameter that determines the evolution of the latent productivity process. Specifically, they believe that it could be high ($\rho_h$) or low ($\rho_l$). At time $t$, the process for the growth rate of the technology shock is given by

$$g_{A_k,t+1} = \bar{g} + x_{k,t+1} + \sigma_{A_k}\epsilon_{A_k,t+1}, \tag{2.4}$$

$$x_{k,t+1} = \rho_k x_{k,t} + \sigma_{x_k}\epsilon_{x_k,t+1}, \tag{2.5}$$

where $(\epsilon_{A_k,t+1}, \epsilon_{x_k,t+1})' \sim N(0, I)$ for $k = h, l$. Here $\bar{g}$ denotes the long run trend value for the technology shock, and the latent process $x_{k,t+1}$ is the temporary deviation. Associated with each persistence parameter, there is also a tuple of volatility parameters $(\sigma_{A,k}, \sigma_{x,k})$. Given these assumptions, next period's technology shock is written as

$$A_{t+1} = A_t \exp(g_{A_k,t+1}) = A_t \exp(\bar{g} + x_{k,t+1} + \sigma_{A_k}\epsilon_{A_k,t+1}). \tag{2.6}$$

According to this representation, the growth rate of the technology shock between $t$ and $t+1$ evolves as a function of a known permanent mean, $\bar{g}$, an unknown temporary component $x_{k,t+1}$, and some noise. At time $t$, the agent has available observations on the current and past values of the technology shock, $A_t$. However, the agent does not know the process generating $x_{k,t}$ and forms beliefs about it, given prior beliefs at time 0 and the observations on $A_t, A_{t-1}, \ldots$ .

## 2.3 Beliefs

The agent is assumed to know the value of the parameters $(\bar{g}, \sigma_{A,k}, \sigma_{x,k})$. The agent also observes contemporaneously the normalized values of consumption, investment, capital stock, and output. Given $x_{k,t}, \rho_k$ and the current of observations or node $(c_t, i_t, k_t, y_t)$, the probability distribution over $g_{A_k,t+1}$ is given by

$$g_{A_k,t+1} \sim N(\bar{g} + \rho_k x_{k,t}, \sigma^2_{A_k} + \sigma^2_{x_k}).$$

This distribution denotes the typical first-order distribution $\pi_\theta(s_{t+1}|s^t)$ in the original KMM formulation.

We now turn to a characterization of second-order uncertainty. The support of the second-order distribution is a union of two component sets, $\{\rho_l x_{l,t} | x_{l,t} \in \Re\}$ and $\{\rho_h x_{h,t} | x_{h,t} \in \Re\}$. The agent's prior belief ascribes a measure to each component set, with the measure on the first component being given by $\eta_0 \times N(0, \sigma_0^2)$ and that on the second component by $(1-\eta_0)\times N(0, \sigma_0^2)$. The agent updates his beliefs using Bayes rule, based on the history of growth realizations and under the assumption that the economy conforms to one of the two processes described above. Let $\hat{x}_{k,t} \equiv E[x_{k,t}|g_{A,1}, \ldots, g_{A,t}]$ denote the expectation of $x_{k,t}$, conditional on the history of growth rates up to $t$ if the beliefs were updated assuming $\rho = \rho_k$ is the data generating process. The filtered latent state, $\hat{x}_{k,t}$, is obtained by applying the (steady state) Kalman filter that takes the process $\rho = \rho_k$ as the "true" data generating process. The agent's posterior beliefs then ascribes a measure on the first component set given by $\eta_t \times N(\hat{x}_{l,t}, \Omega_l)$ and that on the second by $(1-\eta_t) \times N(\hat{x}_{h,t}, \Omega_h)$, where $\Omega_k, k = l, h$ denotes the steady state variance associated with the Kalman filter based on the process with $\rho = \rho_k$ and $\eta_t$ shows the posterior belief on $\rho_l$. Hence, the agent's beliefs may be summarized by the tuple $\mu_t = (\hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t)$.

## 2.4 The social planner's problem

We initially consider the social planner's problem for this economy. Given the stochastic growth in the technology shock, the state variables for the social planner's problem consist of the initial capital $k_t$, initial beliefs $\mu_t$ and the level of the technology shock, $A_t$. Thus, the generic social planner's problem is given by

$$J(k_t, \mu_t, A_t) = \max_{c_t, n_t, i_t} \left\{ u(c_t, l_t) + \beta \phi^{-1} \left[ E_{\mu_t} \phi \left( E_{x_t} J(k_{t+1}, \mu_{t+1}, A_{t+1}) \right) \right] \right\} \tag{2.7}$$

subject to

$$c_t + i_t \leq k_t^a (A_t n_t)^{1-a},$$

$$k_{t+1} = (1-\delta) k_t + i_t,$$

$$l_t + n_t \leq 1,$$

$$i_t \geq 0,$$

and the law of motion for beliefs which we discuss in the next section.[5]

Since the technology shock $A_t$ is nonstationary, we will consider the transformed value func-

[5] It is possible to replicate the solution to the social planner's problem in a recursive competitive equilibrium where the representative consumer makes consumption and labor supply choices and holds shares and bonds in the firm while all production and capital accumulation decisions are made by value-maximizing firms. In this setup, the presence of an irreversibility constraint is equivalent to assuming that there are no resale markets for used capital; see Altug and Labadie (2008), Ch. 10 for a discussion. Also see Kaltenbrunner and Lochstoer (2010) for a similar decentralization scheme in the context of an economy with non-expected utility preferences.

tion in terms of stationary variables. Allowing for an endogenous hours choice, a stationarity inducing transformation exists under two different sets of assumptions for the utility function and the smooth ambiguity aversion function, namely, a power-power specification for risk aversion and ambiguity aversion and log-exponential specification, respectively. These specifications are similar to the cases considered by Ju and Miao (2007).The details of the stationarity-inducing transformations are in the Appendix, Section A.

Here we present the transformed problem for the power-power specification. Define

$$\left\{\hat{c}_t, \hat{i}_t, \hat{k}_t, \hat{y}_t\right\} = \left\{\frac{c_t}{A_t}, \frac{i_t}{A_t}, \frac{k_t}{A_t}, \frac{y_t}{A_t}\right\}$$

. In general, the indirect value function depends on the state variables $k_t, \mu_t$ and $A_t$. However, making use of the homogeneity of the indirect value function, it is straightforward to show that it satisfies

$$J(k_t, \mu_t, A_t) = \hat{J}(\hat{k}, \mu_t) A_t^{(1-\gamma)\nu}.$$

Thus, the stationary version of the indirect value function for the social planner's problem is given by

$$\hat{J}(\hat{k}_t, \mu_t) = \max_{\hat{c}_t, n_t, \hat{i}_t} \left\{ \frac{(\hat{c}_t^{\nu} l_t^{1-\nu})^{1-\gamma}}{1-\gamma} + \frac{\beta}{1-\gamma} \left[ E_{\mu_t} \left( E_{x_t} (1-\gamma) \hat{J}(\hat{k}_{t+1}, \mu_{t+1}) \exp(\nu(1-\gamma) g_{A,t+1}) \right)^{1-\alpha} \right]^{\frac{1}{1-\alpha}} \right\} \tag{2.8}$$

subject to

$$\hat{c}_t + \hat{i}_t \leq \hat{k}_t^a n_t^{1-a},$$

$$\exp(g_{A,t+1}) \hat{k}_{t+1} = (1-\delta) \hat{k}_t + \hat{i}_t,$$

$$l_t + n_t \leq 1,$$

$$\hat{i}_t \geq 0,$$

and the law of motion for beliefs to be discussed below.

### 2.4.1 The optimality conditions

Define the quantities

$$\Upsilon_t = \frac{E_{\mu_t}\left(E_{x_t}\hat{J}(\hat{k}_{t+1},\mu_{t+1})\exp(\nu(1-\gamma)g_{A,t+1})\right)^{-\alpha}}{\left[E_{\mu_t}\left(E_{x_t}\hat{J}(\hat{k}_{t+1},\mu_{t+1})\exp(\nu(1-\gamma)g_{A,t+1})\right)^{1-\alpha}\right]^{\frac{-\alpha}{1-\alpha}}} \tag{2.9}$$

$$\xi_t = \frac{\left(E_{x_t}\hat{J}(\hat{k}_{t+1},\mu_{t+1})\exp(\nu(1-\gamma)g_{A,t+1}))\right)^{-\alpha}}{\left[E_{\mu_t}\left(E_{x_t}\hat{J}(\hat{k}_{t+1},\mu_{t+1})\exp(\nu(1-\gamma)g_{A,t+1})\right)^{-\alpha}\right]}. \tag{2.10}$$

Let $\lambda_t$ denote the Lagrange multiplier on the aggregate resource constraint and $\varphi_t$ the multiplier on the non-negativity constraint. Using the expressions for $\Upsilon_t$ and $\xi_t$, the first-order conditions with respect to $\hat{c}_t, l_t, \hat{i}_t$ are given by

$$\nu(\hat{c}_t^{\nu}l_t^{1-\nu})^{-\gamma}(\hat{c}_t/l_t)^{\nu-1} = \lambda_t, \tag{2.11}$$

$$(1-\nu)(\hat{c}_t^{\nu}l_t^{1-\nu})^{-\gamma}(\hat{c}_t/l_t)^{\nu} = (1-a)\lambda_t\hat{k}_t^{a}n_t^{-a}, \tag{2.12}$$

$$\lambda_t - \varphi_t = \Upsilon_t E_{\mu_t}\left[\xi_t E_{x_t}\left(\hat{J}_1(\hat{k}_{t+1},\mu_{t+1})\exp((\nu(1-\gamma)-1)g_{A,t+1})\right)\right] \tag{2.13}$$

Finally, the envelope condition is given by

$$\hat{J}_1(\hat{k}_t,\hat{\mu}_t) = \lambda_t a\hat{k}_t^{a-1}n_t^{1-a} + (1-\delta)(\lambda_t-\varphi_t). \tag{2.14}$$

The conditions (2.11-2.12) simplify to yield the condition describing the intratemporal substitution in consumption and labor supply

$$\frac{1-\nu}{\nu}\frac{\hat{c}_t}{l_t} = (1-a)\hat{k}_t^{a}n_t^{-a}. \tag{2.15}$$

Given a solution for $\hat{k}_{t+1}$ as a function of $(\hat{k}_t,\mu_t)$, this condition can be solved for current $l_t$ for each $\hat{k}_t$ and beliefs $\mu_t$.

Likewise, defining $\exp(g_{A,t+1})\hat{k}^1_{t+1} = (1-\delta)\hat{k}_t + \hat{i}_t$ for $\hat{i}_t > 0$ and $\exp(g_{A,t+1})\hat{k}^0_{t+1} = (1-\delta)\hat{k}_t$ for $\hat{i}_t = 0$, evaluating the expressions for $\Upsilon_t$ and $\xi_t$ at these expressions, and using the first-order conditions that hold time $t+1$, the conditions describing the optimal choice of investment are given by

$$\lambda_{t+1} = \Upsilon_t^1 E_{\mu_{t+1}}\left[\xi_{t+1}^1 E_{x_{t+1}}\left(\hat{J}_1(\hat{k}_{t+2},\hat{\mu}_{t+2})\exp((\nu(1-\gamma)-1)g_{A,t+2})\right)\right], \hat{i}_{t+1} > 0$$

$$\lambda_{t+1} > \Upsilon_t^0 E_{\mu_{t+1}}\left[\xi_{t+1}^0 E_{x_{t+1}}\left(\hat{J}_1((1-\delta)\hat{k}_{t+1},\hat{\mu}_{t+2})\exp((\nu(1-\gamma)-1)g_{A,t+2})\right)\right], \hat{i}_{t+1} = 0.$$

Now consider a version of the envelope condition that holds at time $t+1$. Using the above results, we obtain

$$\hat{J}_1(\hat{k}_{t+1},\mu_{t+1}) = \lambda_{t+1}\Big\{a\hat{k}_{t+1}^{a-1}n_{t+1}^{1-a} + (1-\delta)\min\Big(1,$$
$$\Upsilon_{t+1}^0 E_{\mu_{t+1}}\left[\xi_{t+1}^0 E_{x_{t+1}}\left(\frac{\hat{J}_1((1-\delta)\hat{k}_{t+1},\hat{\mu}_{t+2})\exp((\nu(1-\gamma)-1)g_{A,t+2})}{\lambda_{t+1}}\right)\right]\Big)\Big\}.$$

The envelope condition provides an expression for the marginal value of capital next period. When there is an irreversibility constraint, the marginal value of capital accounts for the fact that the irreversibility constraint may be binding next period. It is this aspect that leads to an *endogenous risk premium* or an option value to wait in the model with an irreversibility constraint. (For further details, see Demers *et al.* (2003).)

Using these results, the optimal choice of investment at time $t$ can now be written as

$$1 = \Upsilon_t^1 E_{\hat{\mu}_t}\Big\{\xi_t^1 E_{x_t}\Big[\exp(((1-\gamma)\nu-1)g_{A,t+1})\left(\frac{\lambda_{t+1}}{\lambda_t}\right)\Big(a\hat{k}_{t+1}^{a-1}n_t^{1-a} + (1-\delta)\min\Big(1,$$
$$\Upsilon_{t+1}^0 E_{\mu_{t+1}}\left[\xi_{t+1}^0 E_{x_{t+1}}\left(\frac{\hat{J}_1((1-\delta)\hat{k}_{t+1},\hat{\mu}_{t+2})\exp((\nu(1-\gamma)-1)g_{A,t+2})}{\lambda_{t+1}}\right)\right]\Big)\Big)\Big]\Big\}$$
$$\text{if } \hat{i}_t > 0 \tag{2.16}$$

$$1 > \Upsilon_t^0 E_{\hat{\mu}_t}\Big\{\xi_t^1 E_{x_t}\Big[\exp(-\gamma g_{A,t+1})\left(\frac{\lambda_{t+1}}{\lambda_t}\right)\Big(a\hat{k}_{t+1}^{a-1}n_{t+1}^{1-a} + (1-\delta)\min\Big(1,$$
$$\Upsilon_{t+1}^0 E_{\mu_{t+1}}\left[\xi_{t+1}^0 E_{x_{t+1}}\left(\frac{\hat{J}_1((1-\delta)\hat{k}_{t+1},\hat{\mu}_{t+2})\exp(((1-\gamma)\nu-1)g_{A,t+2})}{\lambda_{t+1}}\right)\right]\Big)\Big)\Big]\Big\}$$
$$\text{if } \hat{i}_t = 0. \tag{2.17}$$

These conditions allow us to examine the impact of ambiguity and ambiguity aversion on the optimal choices in equilibrium. In our framework, ambiguity aversion arises from the nature of the processes that generate TFP growth. Such ambiguity aversion leads to an endogenous tilting or distortion of the posterior distributions that signify agent's subjective beliefs about the validity of a given process describing the external environment. The optimality conditions in (2.17-2.16) depend on the distortion factors $\Upsilon_t$ and $\xi_t$. While $\Upsilon_t$ does not affect the second-order distributions appearing in the optimality conditions (2.17-2.16), the distortion term defined by $\xi_t$ does. In this expression, $\xi_t$ depends expectations that are taken with respect to the distribution of $x_t$, conditional on information of the history the technology shock $(g_{A,t}, g_{A,t_1}, \cdots)$ and, hence, is random from the view of the agent's subjective beliefs at date $t$. The function $\xi_t$ may be viewed as the factor that create the endogenous tilting or distortion in agents' beliefs due to ambiguity aversion. In the case of ambiguity aversion with $\alpha > \gamma > 0$, the distortion puts greater weight (relative to a pure Bayesian decision-maker) on the probability distributions of the $x_t's$ associated with lower expected continuation values, $E_{x_t} \hat{J}(\hat{k}_{t+1}, \mu_{t+1}) \exp(\nu(1-\gamma) g_{A_k,t+1}))$. Thus, we may view the impact of the $\xi_t$ as shaping the "as if" beliefs of the agent, that is, the (probabilistic) belief that supports the action chosen in equilibrium. In the absence of ambiguity aversion, the agent behaves as a pure Bayesian decision-maker who is uncertain about the temporary component of TFP growth $x_t$, and has beliefs that are just a mixture of the probability distributions for $x_{kt}, k = l, h$, as discussed above

## 2.5 A Real Business Cycle model with ambiguity aversion

We now turn to evaluation of the model with ambiguity aversion and irreversible investment with the specified beliefs. Denote by $\hat{x}^{(i)}_{k,t+1}, i = l, h, k = l, h$, the agent's forecast for the one-period ahead update using a Kalman filter which takes the model with $\rho = \rho_i$ as the data generating process, when the data is actually generated by the $\rho = \rho_k$ process. Correspondingly, $\eta^{(l)}_{t+1}$ (respectively, $\eta^{(h)}_{t+1}$) is the posterior probability that the low persistence process is the correct model when the low (high) persistence process is the correct model. Under the assumption that consumption, investment and the capital stock normalized by the level of the technology shock, $\hat{c}_t = c_t/A_t, \hat{i}_t = i_t/A_t$, and $\hat{k}_t = k_t/A_t$, are stationary random variables and noting that

$\mu_t = (\hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t)'$, the indirect value function can now be defined as:

$$\hat{J}(\hat{k}_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) = \max_{\hat{c}_t, n_t, \hat{i}_t} \Bigg\{ \frac{(\hat{c}_t^{\nu} l_t^{1-\nu})^{1-\gamma}}{1-\gamma} +$$
$$\beta \Bigg[ \eta_t E_{\hat{x}_{l,t}} \left( E_{x_{l,t}} \hat{J}(\hat{k}_{t+1}^{(l)}, \hat{x}_{h,t+1}^{(l)}, \hat{x}_{l,t+1}^{(l)}, \eta_{t+1}^{(l)}) \exp((1-\gamma) g_{A_l,t+1}) \right)^{1-\alpha} +$$
$$(1-\eta_t) E_{\hat{x}_{h,t}} \left( E_{x_{h,t}} \hat{J}(\hat{k}_{t+1}^{(h)}, \hat{x}_{h,t+1}^{(h)}, \hat{x}_{l,t+1}^{(h)}, \eta_{t+1}^{(h)}) \exp((1-\gamma) g_{A_h,t+1}) \right)^{1-\alpha} \Bigg]^{\frac{1}{1-\alpha}} \Bigg\} \tag{2.18}$$

subject to

$$\hat{c}_t + \hat{i}_t \leq \hat{k}_t^a n_t^{1-a},$$

$$\exp(g_{A,t+1}) \hat{k}_{t+1} = (1-\delta) \hat{k}_t + \hat{i}_t,$$

$$l_t + n_t \leq 1,$$

$$\hat{i}_t \geq 0,$$

and given the laws of motion for beliefs $x_{j,t+1}^{(k)}$, $j = l, h$ together with the condition determining the evolution of $\eta_{t+1}^{(k)}$ for $k = l, h$ described in the Online Appendix, Section B.

In the specification described above, it is worth noting that the agent is ambiguity averse if $\alpha > \gamma$. Another way of understanding the presence of ambiguity aversion is that it precludes the evaluation of future utilities based on the predictive distribution of growth rates. For an ambiguity neutral agent with $\alpha = 0$ who cannot distinguish the high persistence/low variability process from the low persistence/high variability process, the agent's subjective beliefs and the conditional distribution of the technology growth rates reduce to the predictive distributions of a Bayesian learner. In this case, the predictive distributions over future growth rates $g_{A_k,t+1}$ conditional on current and past growth rates $(g_{A,t}, \ldots, g_{A,0})$ are given by $g_{A_k,t+1} \sim N(\bar{g} + \rho_k \hat{x}_{k,t}, \rho_k^2 \Omega_k + \sigma_{x_k}^2 + \sigma_{A_k}^2)$.[6] When ambiguity aversion is present, however, no such reduction is possible.

[6] We could also consider a third case in which the agent knows the distribution from which $g_{A,t}$ is coming from but must make inferences about the unobserved cyclical component, $x_{t+1}$, given current and past observations on productivity growth.

Table 1: Unconditional business cycle moments

| | Full Sample - 1947:1-2015:4 | | | | | Restricted Sample - 1980:1-2015:4 | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | Investment-Output Ratio | | | | | |
| $i/y$ | 0.1425 | | | | | 0.1568 | | | | |
| | | | | | Standard deviations | | | | | |
| | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | $p$ |
| | 1.6244 | 1.2621 | 7.4416 | 1.9249 | 0.9341 | 1.2938 | 1.0755 | 6.2839 | 1.7830 | 0.9296 |
| | | | | | Correlations | | | | | |
| | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | $p$ |
| $y$ | 1.0000 | 0.7742 | 0.8406 | 0.8863 | -0.0640 | 1.0000 | 0.8662 | 0.9127 | 0.8645 | -0.2664 |
| $c$ | | 1.0000 | 0.6943 | 0.6581 | -0.0677 | | 1.0000 | 0.7299 | 0.7678 | -0.2670 |
| $i$ | | | 1.0000 | 0.7615 | -0.0999 | | | 1.0000 | 0.8167 | -0.2962 |
| $h$ | | | | 1.0000 | -0.5393 | | | | 1.0000 | -0.7147 |
| $p$ | | | | | 1.0000 | | | | | 1.0000 |

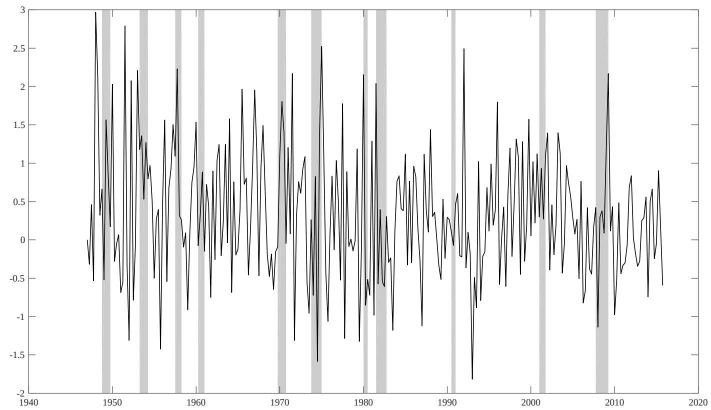
*Note*: This table shows the average investment-output ratio and the unconditional second moments of the HP-filtered series on output, consumption, investment, hours worked, and labor productivity.

# 3 Quantitative results

In Table 1, we present the unconditional business cycle moments for the full sample between 1947:12015:4 and the restricted sample between 1979:1-2015:4. All series are Hodrick-Prescott filtered versions of the original series. The data on output, consumption, investment, and hours worked are obtained from the Federal Reserve Bank of St. Louis database (FRED). The output, consumption, and investment series are seasonally adjusted and measured in chained 2009 dollars. Investment refers to total private investment and the hours worked series is an index of total hours worked in the nonfarm business sector, seasonally adjusted with the 2009 value equal to 100. Table 1 shows that the volatility of the endogenous variables are considerably lower in the restricted sample while the correlations between the different variables are higher. This finding might be taken as evidence of the "Great Moderation" that various authors such as Stock and Watson (2002) have documented for the post-1980's era. We also observe that the correlations of labor productivity $p = y/h$ with output, consumption, and investment are nearly zero for the full sample but these become negative for the restricted sample.

To generate the simulations of the model, we subsequently estimate processes with different

Figure 1: The growth rate of factor utilization adjusted TFP (in percentage terms) over the sample 1947:2-2015:4




persistence parameters that contain a permanent and unobserved transitory component. Hence, we also discuss the cyclical behavior of total factor growth for the full sample and the post-1980 sample. We use seasonally adjusted data on total factor productivity (TFP) growth obtained from the Federal Reserve of San Francisco, see Fernald (2012) for details. The data on inputs, including capital are used to produce a real-time, quarterly series on total factor productivity growth as the measured Solow residual. An additional advantage of these data are that they are adjusted to account for the changes in factor utilization. Figure 1 displays the growth rates of the factor utilization adjusted TFP series measured at annual rates for the full sample together with NBER recession dates.

Recently, many authors have observed that there has been a secular decline in TFP growth; see Gordon (2015). This finding is also evident from Figure 1. Specifically, average TFP growth has declined in the post-1980's relative to the pre-1980's period from 1.78% to 0.88% for the adjusted TFP growth series, measured at annual rates. There is also a decline in the variability of TFP growth after 1980 but this is not as great as the decline in the average quantities. Specifically, the standard deviation of adjusted TFP growth has fallen from 3.75% in the pre-1980's period to to 2.94% in the post-1980's period.[7]

[7]Similar findings hold for the unadjusted TFP growth series. Specifically, its growth rate has declined from

## 3.1 Bayesian estimation of the underlying TFP process

The model for the TFP growth process to be estimated is as follows:

$$\begin{aligned} g_{A,t+1} &= \bar{g} + x_{t+1} + \sigma_{A,k}\epsilon_{A,t+1} \\ x_{t+1} &= \rho_k x_t + \sigma_{x,k}\epsilon_{x,t+1} \end{aligned} \tag{3.1}$$

In the Online Appendix, Section C, we describe the estimation of the model's parameters based on an unrestricted model using simulation based Bayesian inference with noninformative priors. As the results reported there show, the distribution of the persistence parameter $\rho$ covers a wide range of values between 0 and and 0.90 with high probability. This also indicates the difficulty to pinpoint the exact value of the persistence in the TFP growth process. To see this further, next, we estimate the models for given values of $\rho$, i.e. $\rho = 0.25, 0.30, 0.65, 0.70, 0.85, 0.90$. The results are displayed in Table 2.

The results in Table 2 confirms our findings on the (in)-ability to identify the exact value of the persistence parameter. Regardless of the values of the $\rho$ ranging from 0.20 to 0.90, the value of maximum likelihood only changes after the second digit and the change in the marginal likelihoods are very minor. In the Online Appendix, Section C, we also display the distribution of the parameters where we set the values of $\rho$ as 0.30 and 0.85 for the low and high persistence cases, respectively.

Given the parameter setup indicated in Table 2, we can also compute agent's belief on the true DGP denoted $\eta_t$ for the low persistence model with $\rho = 0.30$ relative to the one with $\rho = 0.85$. This can be computed using the agent's updating mechanism after observing data on the actual TFP process. We compute the sequence of $\eta_t$'s over the period starting from 1947:12 until 2015:4 using the parameter setup as shown in Table 2. We use the steady state Kalman filter to compute the beliefs. In line with the results so far, the probabilities attached to each separate process vary in a band around 0.50-0.55, with some tendency to increase above this value towards the end of the sample. This suggests that there is very little learning that is occurring over the sample period, though we do see an increase in the probability attached to

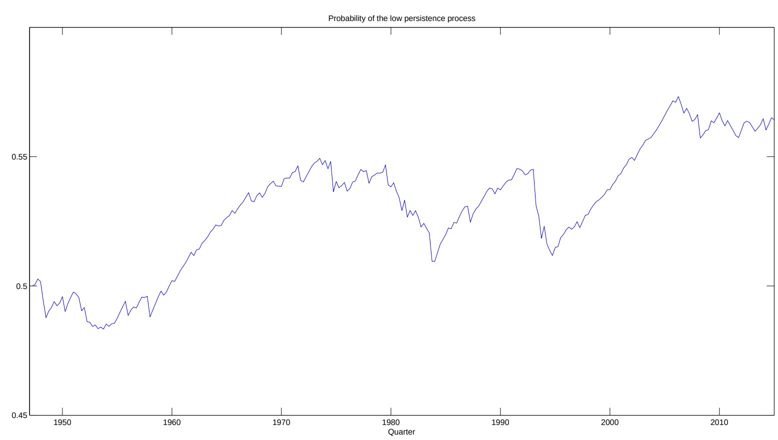
1.72% to 0.85% in the post-1980's relative to the pre-1980's while its standard deviation has declined from 4.07% to 2.78% across the two periods.

Table 2: Posterior results for the model using TFP-util for different values of $\rho$ using the sample of 1947-2 : 1977-4

| | 0.25 | 0.30 | 0.65 | 0.70 | 0.85 | 0.90 |
|---|---|---|---|---|---|---|
| $\bar{g}$ | 0.469 (0.086) | 0.469 (0.086) | 0.469 (0.086) | 0.469 (0.086) | 0.469 (0.086) | 0.469 (0.086) |
| $\sigma_g$ | 0.945 (0.075) | 0.946 (0.071) | 0.949 (0.063) | 0.950 (0.063) | 0.952 (0.062) | 0.953 (0.062) |
| $\sigma_x$ | 0.046 (0.120) | 0.044 (0.110) | 0.056 (0.080) | 0.054 (0.073) | 0.040 (0.049) | 0.033 (0.040) |
| $\rho$ | 0.250 | 0.300 | 0.650 | 0.700 | 0.850 | 0.900 |
| Max. | -167.40 | -167.40 | -167.40 | -167.40 | -167.40 | -167.40 |
| Mar. | -169.70 | -169.73 | -170.16 | 170.20 | -170.32 | -170.38 |

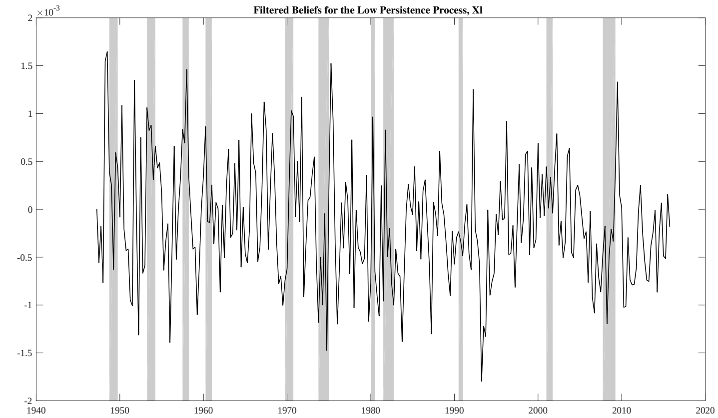
*Note*: The results show the posterior means and posterior standard deviations (in parenthesis) of the model parameters in (3.1) (evaluated in percentage terms). The inference was carried out with 60,000 draws where the first 10,000 are used as burn-in sample. We kept every $5^th$ draw, which yields a sample of 10,000 draws from the ergodic distribution.

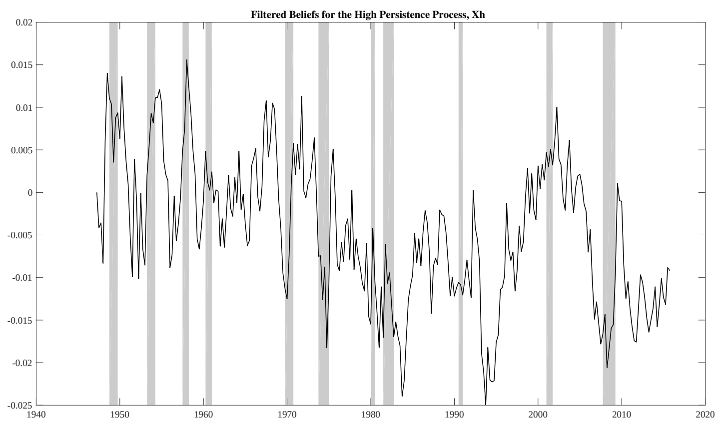
Figure 2: Evolution of the probability for the low persistence model with $\rho = 0.30$ to be the true DGP over the sample 1947:2:2015-4




the low persistence process in the run-up to the 2008 global financial crisis. The two panels Figure 3 further show the time series of the filtered means, $\hat{x}_{l,t}$ and $\hat{x}_{h,t}$ estimated using data on actual TFP growth. These figures show that the filtered means tend to decline during the recessions of the 1970's and 1980's as well as during the global financial crisis of 2008. This decline is particularly severe for agents' beliefs regarding the cyclical mean of the high persistence process, $\hat{x}_{ht}$.

Figure 3: Filtered means (in percentage terms) conditional on the processes with $\rho = 0.30$ and $\rho = 0.85$ over the sample 1947:2:2015-4




In the Online Appendix, we also allow for informative priors and generate the sequence of filtered means and the mixing probabilities denoted by $\hat{x}_{lt}, \hat{x}_{ht}, \eta_t$ associated with the estimated models under those priors. As expected, even slightly informative priors are associated with an eventually higher probability being placed on low persistence process. We will discuss the ramifications of these estimation results for the model's behavior in the next section.

### 3.2 Simulation results

In this section, we calculate the unconditional moments for all of the series by randomly drawing sample of shocks from the high and low persistence processes to generate a pseudo observation on the growth rate of technology $g_{A,t+1}$ at each date. We continue to assume that the agent does not know which process the realization of the technological growth shock is coming from and must make inferences from observations of the growth rate of the technology process about the nature of process form which such observations are drawn. Given initial conditions $\hat{k}_0 = k_s, \hat{x}_{l0} = 0, \hat{x}_{h0} = 0, \eta_0 = 0.5$, we use the laws of motion for the capital together with the Kalman filtering algorithm to determine the evolution of the capital stock and beliefs along the agent's optimal path. These will constitute the endogenous state variables for the model. Notice that updating the capital stock depends on using the optimal policy functions for investment and hours worked as $\hat{i}_t = g(\hat{k}_t, \hat{x}_{lt}, \hat{x}_{ht}, \eta_t)$ and $h_t = h(\hat{k}_t, \hat{x}_{lt}, \hat{x}_{ht}, \eta_t)$ evaluated at the current state $(\hat{k}_t, \hat{x}_{lt}, \hat{x}_{ht}, \eta_t)$. Given the simulated values of $\hat{i}_t, h_t$ and $g_{A,t+1}$, output and consumption today together with next period's capital stock are obtained from the production function, the resource constraint and the law of motion for capital as $\hat{y}_t = (\hat{k}_t)^a h_t^{1-a}$, $\hat{c}_t = \hat{y}_t - \hat{i}_t$, and $\hat{k}_{t+1} = (\hat{i}_t + (1-\delta)\hat{k}_t)\exp(-g_{A,t+1})$. To generate next period's beliefs which will form next period's state vector, we update current beliefs $\mu_t = (\hat{x}_{lt}, \hat{x}_{ht}, \eta_t)$ given the new observation $g_{A,t+1}$ using the Kalman filtering algorithm.[8]

[8]Specifically, the filtered beliefs about the cyclical component conditional on the $k$'th persistence process are then given by

$$\hat{x}_{k,t+1} = \rho_k \hat{x}_{kt} + K_k \nu_{k,t+1}, \quad k = l, h,$$

where the Kalman "surprises" $\nu_{k,t+1}$ are given by $\nu_{k,t+1} = g_{A,t+1} - \bar{g} - \rho_k \hat{x}_{kt}$, and the Kalman gain parameters conditional on the $k$th persistence process being the true process are given by $K_k = \rho_k \Omega_k f_k^{-1}, \quad f_k = \Omega_k + \sigma^2_{A_k}$, where $f_k = E[(g_{A_k,t+1} - E(g_{A_k,t+1}))^2 | g_{A,1}, \ldots, g_{A,t}]$, and $\Omega_k = E[(x_{k,t+1} - \hat{x}_{k,t+1})^2 | g_{A,1}, \ldots, g_{A,t}]$, $k = l, h$ are defined as the solution to

$$\Omega_k = \rho_k^2 \Omega_k - \rho_k^2 \Omega_k^2 f_k^{-1} + \sigma^2_{x_k}.$$

The Bayes update $\eta_{t+1}$ shows the posterior belief that the $\rho_l$ process is the true one and it is given by

$$\eta_{t+1} = \frac{\eta_t L(\nu_{l,t+1}, f_l)}{\eta_t L(\nu_{l,t+1}, f_l) + (1-\eta_t) L(\nu_{h,t+1}, f_h)},$$

where $L(\nu_{k,t+1}, f_k) = \frac{1}{\sqrt{(2\pi f_k)}} \exp\left(-\frac{-\nu^2_{k,t+1}}{2 f_k}\right)$.

Table 3: Parameter Values

| | | |
|---|---|---|
| $\beta$ | Subjective discount factor | 0.988 |
| $\gamma$ | Coefficient of ris k aversion | 0.5, 2 |
| $\alpha$ | Coefficient of ambiguity aversion | 0.8, 2, 5 |
| $a$ | Capital share | 0.3 |
| $\delta$ | Depreciation rate | 0.025 |

*Note*: This table reports parameter choices used in the simulation of the model variations. The calibration is at the quarterly frequency. The share of leisure is implicitly computed from the intratemporal marginal rate of substitution between consumption and leisure, assuming a steady value of hours worked $n_{ss} = 1/3$.

The parameter values used in the simulations are standard to the business cycle literature. Specifically, the capital share is set at 0.36 and the depreciation at the quarterly frequency at 0.025. The share of leisure in preferences denoted by the parameter $\nu$ is given by 0.3663. This is based on the steady state values of the model where the share of working time is set at 1/3, consistent with the finding that households spend one-third of their time working. The discount rate is set at $\beta = 0.988$, which is slightly lower than the value assumed by Prescott (1986) and implies an annual interest rate of around 4.8%. The estimates of Table 2 imply that the average growth rate of the technology shock at the quarterly frequency is 0.46%, which is slightly higher but consistent with the average growth rate of technology based on the Markov switching model reported in Jahan-Parvar and Liu (2012) as well as earlier estimates in Kaltenbrunner and Lochstoer (2010) and Croce (2010).

In Table 4, we display results obtained by simulating the model with ambiguity and ambiguity aversion under alternative technology shock processes. The decision rules for the transformed problem are used to generate levels for the nonstationary series, which are then detrended using a Hodrick-Prescott filter for quarterly series. The simulated moments are based on a 1,000 simulations of 400 periods, with a burn-in sample of 200 periods. In these simulations, the estimates of the technology shock processes are obtained by using uninformative priors in their estimation, as reported in Table 2. While the simulations are done separately for the low persistence/high variance and high persistence/low variance processes by drawing realizations of the shocks from these processes, the agent is assumed not to know from which process the underlying technology shocks is coming from.

The choice of the coefficient of risk aversion and of ambiguity aversion are of importance in their own right. In the smooth model of ambiguity, ambiguity aversion is inferred from properties of the functions $\phi(\cdot)$, and is measured by the parameter $\alpha$ for the specification of preferences used in this paper.[9] Risk aversion, as usual, is inferred from the properties of the $u(\cdot)$, and is measured by the parameter $\gamma$. Notice, however, that the specification of preferences employed here does not allow for a separation of risk aversion and the elasticity of intertemporal substitution (IES). Using the general representation in (2.1) evaluated under deterministic consumption paths, it is straightforward to show that $IES = 1/\gamma$. In our quantitative analysis, we employ the parameter configurations of $\gamma = 0.5, \alpha = 0.8$, $\gamma = 0.5, \alpha = 0.5$, $\gamma = 0.5, \alpha = 5$, and $\gamma = 2, \alpha = 5$ to control separately for the impact of $\gamma$ and $\alpha$.[10]

Table 4 reports the results for the main model with ambiguity and learning. We find that the model is, in general, able to replicate standard business cycle facts. Here (i) the volatility of investment is nearly three times that of output, (ii) output growth is more variable than consumption, and (iii) the magnitudes of the simulated standard deviations match the standard deviations of the actual series for the post-1980's sample period, especially for the case with $\gamma < 1$. However, the model does much more poorly in matching the variability of hours, which is a standard result in the business cycle literature; see Altug (1989) or Hansen (1985). The correlations of the simulated series capture the pro-cyclicality of consumption and investment, though the correlations of the simulated series are lower than the data-based correlations.

In the second panel of Table 4, we increase the value of $\alpha$ to 5, holding the value of $\gamma$ constant at 0.5. In this case, under the assumption that the TFP process is drawn from the high persistence/low variability process, the variability of series such as investment, hours, and

[9]More generally, Klibanoff *et al.* (2005) show that ambiguity aversion is defined as an aversion to mean preserving spreads in the distribution of expected utilities induced by agent's prior beliefs under a specific action, which corresponds to $-\phi''(x)x/\phi'(x)$.

[10]Jahan-Parvar and Liu (2012) and Liu and Zhang (2015) discuss the choice of the ambiguity aversion parameter using the approach of error detection probabilities proposed by Hansen (2007).

output tend to increase slightly relative to the case with $\gamma = 0.5, \alpha = 0.8$ but they tend to fall if the TFP process is drawn from the low persistence/high variability process. Thus, we find that increasing the coefficient of ambiguity aversion, while holding constant the value of $\gamma$, tends to increase the difference in the response of the endogenous series depending on whether the TFP process is drawn from the high persistence/low volatility versus the low persistence/high volatility process, without changing the overall magnitudes significantly compared to the case with $\gamma < 1, \alpha = 0.8$. Thus, under greater ambiguity aversion, agents engage in precautionary saving behavior by smoothing their responses under the low persistence/high volatility process relative to the high process/volatility process. The precautionary behavior of agents under ambiguity aversion is also noted by Cagetti *et al.* (2002), Jahan-Parvar and Liu (2012) and Liu and Zhang (2015), amongst others.

Finally, in the third panel of Table 4, we increase the parameter $\gamma$ to 2, keeping the value of $\alpha$ constant at 5. We now observe that the standard deviations of the investment and hours series fall by nearly 50% and the correlations among the different series increase substantially. Smaller declines occur in the variability of the output series, while the volatility of consumption increases slightly. These findings suggest that it is changes in the parameter $\gamma$ that have a more significant impact on the results compared to changes in the parameter $\alpha$. A similar finding is reported by Liu and Zhang (2015).[11]

To understand the reasons why this occurs, we refer back to the analysis of Tallarini (2000) and Backus *et al.* (2014) regarding the role of uncertainty aversion in real business cycle models of the type studied here. First, although an increase in $\alpha$ is associated with an increase in ambiguity aversion, their analysis suggests that this factor tends to have a minor impact on the behavior of quantities if agents can optimally respond to such uncertainty through their choice of investment and hours to smooth their consumption over time. In the Online Appendix, Section D, we generate a log-linear approximation to the model under the assumption of known

[11] In their sensitivity analysis, they show that changing the value of their ambiguity aversion parameter, $\eta$, has only a minor effect on the volatilities of consumption and investment. Specifically, they find that for $\eta = 30$, the benchmark case, the standard deviations of consumption and investment are given by 1.30 and 5.13, respectively. Increasing the value of $\eta$ to 40 reduces these volatilities as 1.35 and 4.89 while reducing the value of $\eta$ to 20 increases the relevant volatilities as 1.26 and 5.37, respectively.

persistence ($\eta_t = 0$) for the TFP growth process. There we show that the dynamics of the capital stock is unaffected by the ambiguity aversion parameter, $\alpha$, as are the coefficients on the filtered mean of the temporary component of TFP growth, $\hat{x}_t$.[12] These results do not carry over directly to the more general version of the model, as ambiguity continues to play a role in the long-run in that case. When $\eta_t > 0$ so that learning is never complete, the long-run of the economy must be described by the stationary distribution for the capital stock, which we have not characterized. Nevertheless, we would expect the result obtained under the assumption of known persistence ($\eta_t = 0$) to provide insight into the role of ambiguity aversion even in the case with $\eta_t > 0$.[13] Another way of understanding the results in Table 4 is to note that while increasing the value of $\gamma$ leads to an increase in risk aversion, it corresponds to a decline in the EIS. Thus, the impact of increasing $\gamma$ is to make consumers *less* willing to substitute consumption across periods. This factor leads to the decline in the volatility of the endogenous series as well as an increase in their co-movement. Thus, the results in the third panel of Table 4 demonstrate that the impact of reducing the elasticity of intertemporal substitution dominates the impact of increasing agents' ambiguity aversion.

These two interrelated results provide insight into the role of intertemporal substitution versus ambiguity aversion in a Real Business Cycle Model with ambiguity and ambiguity aversion. Moreover, these results do not derive from the form of the preferences or the structure of shocks assumed to drive the business cycle. Indeed, even in the three-parameter specification of preferences and a Markov-switching model for the shocks used by Jahan-Parvar and Liu (2012) and Liu and Zhang (2015), neither risk nor ambiguity aversion have large impacts on the variabilities or correlations involving the endogenous quantity series. Instead, it is changes in the IES that induce changes in the behavior of aggregate quantities. Indeed, both Jahan-Parvar

[12]Our approach follows Backus *et al.* (2014), who perform a log-linear approximation for the solution of the solution planner's problem in the non-expected utility and smooth ambiguity cases without learning.

[13]As Tallarini (2000) and Backus *et al.* (2014) have argued, this result reflects the full insurance/complete markets assumption that underlies the social planner's problem used to generate the business cycle observations. To demonstrate this result, it is possible to formulate a recursive complete contingent claims equilibrium which supports the allocations in the social planner's problem. In this equilibrium, consumers can insure against the future state as well as the distribution from which that state is drawn. We omit this discussion for brevity's sake; details can be found in Altug (2017).

and Liu (2012) and Liu and Zhang (2015) match the business cycle moments by assuming an IES of 2. In our case, as the parameter $\gamma$ is increased from 0.5 to 2, agent's incentive to substitute consumption across different periods decreases, which tends to induce greater consumption smoothing across periods. Initially, Kydland and Prescott (1982) argued that attitudes towards intertemporal substitution in consumption and leisure combined with time lags in investment and stochastically varying technology shocks could, to a first approximation, generate observed aggregate fluctuations in output, consumption, investment, and to a lesser extent, hours worked. Our results show that this finding remains even in a framework that allows for ambiguity and ambiguity aversion in a theoretically consistent manner, and that incorporates an investment friction through an irreversibility constraint.[14]

As another gauge of the impact of ambiguity aversion, we also examine the behavior of the investment-output ratio. This is close to the value reported for the data, and suggests that the model is able to reconcile the average value of 0.1568 for the post-1980's. We observe that the investment-output ratio is higher for simulations based on draws of the low persistence/high volatility process: since good realizations of the technology shocks are not expected to persist, agents form a buffer by increasing their investment relative to output to smooth their consumption over time. We also observe from the bottom panels of Table 4 that due to this phenomenon output, consumption, investment, and hours worked all become less variable despite the greater variability of the technology shock process. Similarly, the investment-output ratio tends to increase for the simulations reported in the middle panel of Table 4, as greater ambiguity aversion leads agents to engage in greater precautionary savings behavior. Jahan-Parvar and Liu (2012) also emphasize the precautionary saving motive implied by ambiguity aversion in their impulse response analysis, but the simulated values of the investment-output ratio for the specifications that they consider are considerably higher than in the data, 0.30 on average in the simulations compared to 0.19 in the data. They attribute this to the high capital share, which they set at 0.36 to match the relative volatilities of output and consumption. By contrast, we are able to

[14]Other production-based models with ambiguity aversion such Jahan-Parvar and Liu (2012) and Liu and Zhang (2015) employ adjustment costs in investment. In separate calculations, we verified that our results are invariant to the inclusion of adjustment costs in place of investment irreversibility.

Figure 4: Response to One Std. Dev. Shock to $\epsilon_{x_k,t}, k = l, h$ - Baseline Model

$\gamma = 0.5, \alpha = 0.8$ $\qquad\qquad$ $\gamma = 2, \alpha = 5$

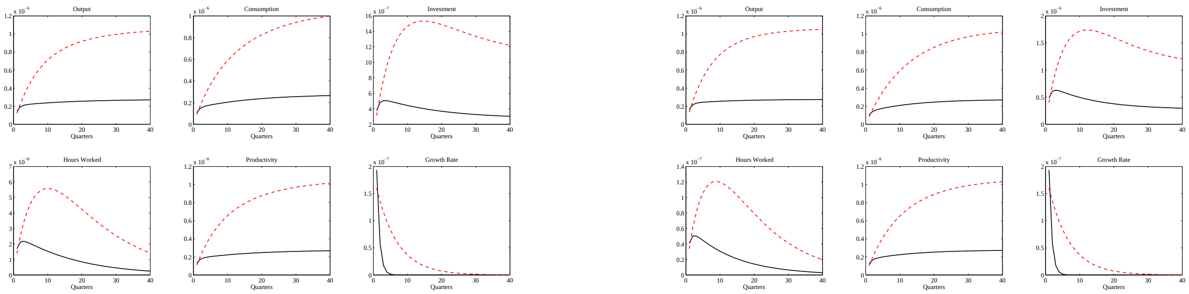
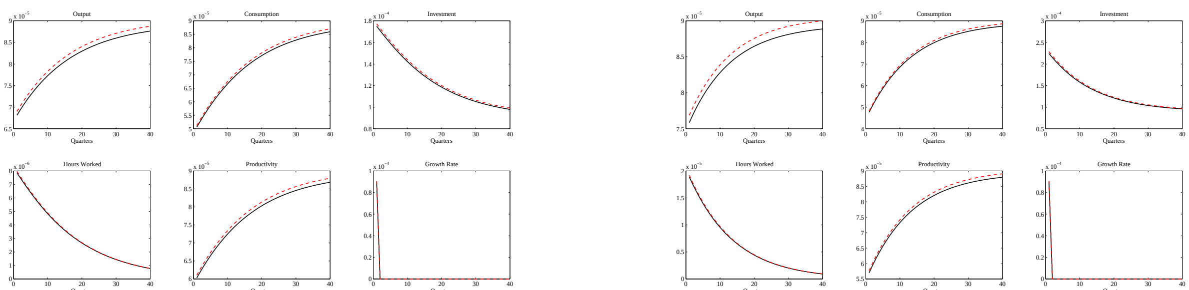



Figure 5: Response to One Std. Dev. Shock to $\epsilon_{A_k,t}, k = l, h$ - Baseline Model

$\gamma = 0.5, \alpha = 0.8$ $\qquad\qquad$ $\gamma = 2, \alpha = 5$




match these volatilities with a much smaller capital share when we set the IES to be greater than 1 ($\gamma < 1$).

Finally, we characterize the behavior of the model by examining the nonlinear impulse response functions developed by Koop *et al.* (1996). We consider the response of each endogenous variable to a one standard deviation shock to $\epsilon_{x_k,t}$ and $\epsilon_{A_k,t}$, $k = l, h$. As before, the decision rules for the transformed problem are used to generate levels for the nonstationary series, which are then detrended using a Hodrick-Prescott filter for quarterly series. We generate the nonlinear IRF's for 40 periods, with a burn-in sample of 200 periods, and consider the average response over 1000 simulations. The case with $\gamma = 0.5, \alpha = 0.8$ case versus the case with $\gamma = 2, \alpha = 5$ are graphed in the top and bottom two panels, respectively. In Figures 4 and

5, the 'red' line corresponds to the generalized IRF for the high persistence economy and the 'black' line for the low persistence economy. Since a shock to the cyclical component of the technology shock for the high persistence economy $\epsilon_{x_{h,t}}$ has a longer lasting impact on technological growth, the responses of the variables in the high persistence economy are always larger: output, consumption, and productivity increase on impact and converge to higher levels monotonically compared to the low persistence economy while the response of investment and hours is hump-shaped, initially rising and falling thereafter. By contrast, the impact of shocks to the cyclical component of technology $\epsilon_{x_{l,t}}$ for the low persistence economy have much more short-lived and hence, muted effects. On the other hand, there is very little difference in the response to shocks to $\epsilon_{A_k,t}$ for the high versus low persistence economies. Since the estimated standard deviations for $\epsilon_{A_l,t}$ and $\epsilon_{A_h,t}$ are very close, this is reflected in the response to such shocks. In this case, a temporarily high growth rate of productivity leads to monotonic increases in output, consumption, and labor productivity while hours and investment go up on impact but fall to lower levels later.

The IRF's for the $\epsilon_{x_k,t}$ and $\epsilon_{A_k,t}$ shocks look very similar to the case with $\gamma = 2, \alpha = 5$. However, there are some differences in the magnitudes of the responses, albeit small. For the case with $\gamma = 2, \alpha = 5$, investment and hours worked show a larger initial response relative to the case with $\gamma = 0.5, \alpha = 0.8$ in response to a $\epsilon_{A_k,t}$ shock while the response of consumption and output are similar. With a lower IES and greater ambiguity aversion under the former case, investment and hours worked tend to rise more in response to a positive productivity shock to maintain the similar consumption and output levels under both cases. Nevertheless, our analysis shows that the variation in the unobserved component, $x_{t+1}$, is small, and the behavior of the different components of TFP growth are similar for the low versus high persistence processes. This is evident, for example, in the impulse responses with respect to the $\epsilon_{A_l,t}$ and $\epsilon_{A_h,t}$ shocks.

# 4 Information and learning

One of the issues that the above analysis has to with the informativeness of the observed TFP series for the cyclical component of productivity, $x_{k,t+1}$. Since ambiguity aversion derives from the nature of this process estimated under alternative scenarios, it is important to examine the sensitivity of our results to alternative estimates of the TFP process. We now consider some alternative experiments regarding the TFP process, which are intended to illustrate the dynamics of learning under ambiguity aversion about the postulated TFP processes.

## 4.1 Known persistence

In Table 5, we initially conduct a counterfactual experiment by assuming that the agent knows for sure the process generating the unobserved component of TFP growth, namely, which process the persistence and variance parameters $(\rho_k, \sigma^2_{x_k})$ are coming from, even if they do not observe the underlying cyclical component $x_{k,t}$ and must infer it using observations on TFP growth up to time $t$. We consider simulations separately with a single process with known persistence equal to $\rho = 0.30$ or $\rho = 0.85$, and continue to allow for non-zero values of $\alpha$ as in Table 4. Here we find that when the agent knows the true process governing the behavior of the cyclical component of TFP growth, his/her behavior tends to exhibit much lower volatility than when he/she does not know the true process. When agents are unsure about the process governing the cyclical component, output and investment becomes substantially more variable and consumption less. By contrast, in the versions of the model with a single known process for the cyclical component, agents are able to act on this knowledge and hence, tend to smooth their optimal investment and hours choices, which also leads to less fluctuations in output and consumption. These results suggest that there is a negative relationship between "confidence" and volatility in our ambiguity model, which reflects the endogenous response of economic agents to additional knowledge about underlying sources of cyclical fluctuations.

## 4.2 Informative priors

To have a more refined view of the impact of information and learning, we also compare the business cycle moments generated by the model under estimates of the TFP process using uninformative priors to those generated by using estimates obtained with more informative priors. We may view more informative prior distributions as a measure of the agent's "confidence" regarding the nature of the underlying TFP process, as described in Appendix D.

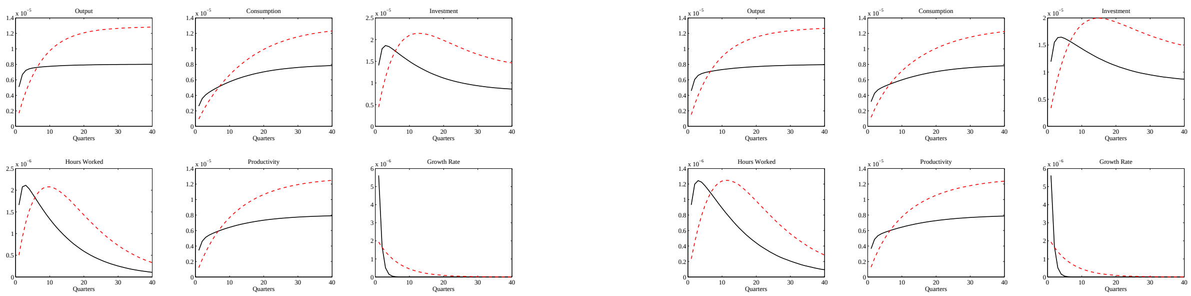
The simulation results in Table 6 based on the estimates on the estimates of Panels A and B of Table C.3 in the Appendix show that when agents are *less* sure about the process governing the cyclical component of TFP, as in Panel A, optimal investment and hours choices become more variable while the variability of output does not change substantially. By contrast, in the versions of the model with a more informative priors in the estimation of the temporary component of the underlying TFP process, as in Panel B, agents are able to act on this knowledge and hence, tend to smooth their optimal investment and hours choices, which also leads to lower fluctuations in output, investment and hours worked but slightly higher volatility for consumption as the residual series. These conclusions are, in general, valid when the technology process is drawn from the high persistence/low variability or the low persistence/high variability technology process and under the alternative parameter sets, $\gamma = 0.5, \alpha = 0.8$ versus $\gamma = 2, \alpha = 5$.[15]

We can also examine the behavior of the investment-output ratio under these parameter configurations. When agents know the process generating realizations of the unobserved cyclical component of TFP growth, the average level of the investment-output ratio is higher compared to the situation in which the nature of these processes must also be inferred based on the past history of observed TFP growth. Thus, we find that lack of information about fundamentals in the economy tends to *depress* average investment as a fraction of GDP, as reported in Table 6, compared to the full information case reported in Table 5. This occurs regardless of whether the TFP shocks are drawn from the high persistence/low volatility versus the low/persistence/high volatility economy. Second, the investment-output ratio is lower, the higher the value of $\gamma$ (or, the lower the EIS), suggesting that the average level of investment relative to GDP responds to consumers' willingness to substitute consumption across different periods. Finally, as in Table 4, we observe that changes in the informativeness of the underlying TFP series have the effect of reducing $i/y$ in the high persistence/low economy economy while increasing it in the

[15]However, for the case with $\gamma = 2, \alpha = 5$, conditional on the observations being drawn from the process with $\rho_k = 0.30$, we observe that the variability of output, investment, and consumption increases slightly under the Panel B estimates with more informative priors.

Figure 6: Response to One Std. Dev. Shock to $\epsilon_{x_k,t}$, $k = l, h$ - Simulations for Panel A Estimates

$\gamma = 0.5, \alpha = 0.8$ $\gamma = 2, \alpha = 5$




low persistence/high volatility economy, suggesting the use of investment/savings as a buffer against variations in the underlying TFP process.

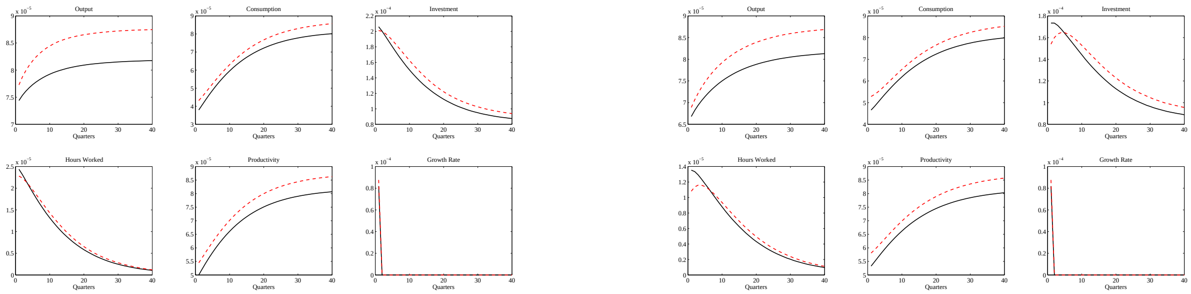
Finally, we report the impulse response functions for the Panel B estimates for the $\gamma = 0.5$, $\alpha = 0.8$ and $\gamma = 2$, $\alpha = 5$ cases.[16] Figure 6 shows that the responses to a one-standard deviation shock to $\epsilon_{x_k,t}$ are more differentiated for low versus high persistence processes (shown in 'black' versus 'red' in the figure) compared to the baseline case displayed in Figure 4. This occurs because a more informative prior on the low persistence process leads to greater differentiation in the standard deviation of the shocks. The differences in the impulse responses are especially pronounced for the $\gamma = 0.5, \alpha = 0.8$ case compared to the one for $\gamma = 2, \alpha = 5$, implying that a lower intertemporal substitution elasticity in the latter case tends to mitigate the response of the quantity series. Figure 7 shows that for the case of the $\epsilon_{A_k,t}$ shocks, there is less difference in the impulse responses precisely because there is less differences in the estimated standard deviations between the high and low persistence processes. One difference between the response of output for the $\gamma = 0.5, \alpha = 0.8$ versus the $\gamma = 2, \alpha = 5$ cases is that output initially rises faster and declines to a lower level in the former case. This may have to do with the behavior of hours worked, which also initially rises faster in response to a positive productivity under the former case but also tends falls to a lower level.

This discussion shows that information and learning effects generate additional transmission

[16]The impulse responses for the Panel A estimates were not substantially different; hence, we omit them for brevity's sake.

Figure 7: Response to One Std. Dev. Shock to $\epsilon_{A_k,t}, k = l, h$ - Simulations for Panel A

$\gamma = 0.5, \alpha = 0.8$ $\gamma = 2, \alpha = 5$




mechanisms under ambiguity and ambiguity aversion that reflect the endogenous response of economic agents to additional knowledge about underlying sources of cyclical fluctuations. This is a transmission mechanism that is missing from standard business cycle models, and it arises from the role of ambiguity and ambiguity aversion in driving agents' actions. Furthermore, unlike the framework of Ilut and Schneider (2014) where ambiguity and ambiguity aversion are modeled in terms of the behavior of the worst case mean of technology, our framework allows a separate treatment of the effects of ambiguity through information and learning about two separate processes versus ambiguity aversion through agents' attitudes towards unknown lotteries.

# 5 Measuring uncertainty and ambiguity aversion: The risk-free rate

In his original analysis, Tallarini (2000) showed that in the type of environments with full risk sharing that we have considered here, uncertainty aversion will tend to manifest itself in asset prices such as the risk-free rate (see also Backus *et al.* (2014)) whereas variation in quantity variables will reflect the effects of intertemporal substitution, as in our analysis above. In this section, we characterize the behavior of the risk-free rate, and argue that it may serve as an endogenous measure of ambiguity implied by our model. As we show below, the risk-free rate tends to reflect the distorted beliefs impled under ambiguity aversion: it declines with the distorted mean and increases with the distorted variance of the posterior distribution regarding the unknown cyclical component of TFP growth. Since the role of ambiguity aversion is to accentuate any increases in exogenous uncertainty regarding the cyclical component of TFP growth, we may view the risk-free rate as reflecting the endogenous response to such changes. As further evidence in this regard, we also present results for the risk-free rate using sectoral TFP data available from Bloom *et al.* (2012), and compare our implied ambiguity measures with their measures of uncertainty at the sectoral level.

## 5.1 The risk-free rate

Using the notation in Section 2.5, the (gross) risk-free rate $R^f$ for the model with ambiguity aversion satisfies

$$1 = \beta R^f \Upsilon_t E_{\mu_t} \left\{ \xi_t E_{x_t} \left[ \exp(((1-\gamma)\nu - 1) g_{A,t+1}) \left( \frac{\lambda_{t+1}}{\lambda_t} \right) \right] \right\}, \tag{5.2}$$

where $\Upsilon_t$ and $\xi_t$ are defined by equations (2.10) and (2.9) and $\lambda_t = \nu(\hat{c}_t^{\nu} l_t^{1-\nu})^{-\gamma}(\hat{c}_t/l_t)^{\nu-1}$. Recall that $\mu_t$ accounts for the second-order beliefs of agents. We can re-write this by making explicit use of the second-order distribution as

$$\begin{aligned} 1 = \beta R^f \Upsilon_t \Bigg\{ & \eta_t E_{\hat{x}_{l,t}} \left[ \xi_t^{(l)} E_{x_{l,t}} \left( \exp(((1-\gamma)\nu - 1) g_{A_l,t+1}) \frac{\lambda_{t+1}^{(l)}}{\lambda_t} \right) \right] \\ & + (1-\eta_t) E_{\hat{x}_{h,t}} \left[ \xi_t^{(h)} E_{x_{h,t}} \left( \exp(((1-\gamma)\nu - 1) g_{A_h,t+1}) \frac{\lambda_{t+1}^{(h)}}{\lambda_t} \right) \right] \Bigg\}. \end{aligned} \tag{5.3}$$

In what follows, we derive a log-linear approximation to the risk-free interest rate under the assumptions in Collard *et al.* (2012), first, by considering the case with known persistence ($\eta_t = 0$) for the growth rate of the TFP process and second, by treating the distorted or "as if" posterior $\tilde{\mu}_t \equiv \xi_t(x_t) \otimes N(\hat{x}_t, \Omega)$, where the distortion arises from the role of ambiguity aversion, as a normal density with variance $\Omega$ but a different mean, $\tilde{x}_t$. Using the assumptions, let $E_t \equiv E_{\hat{x}_t} E_{x_t}$ and $\tilde{E}_t \equiv E_{\tilde{\mu}_t} E_{x_t} \equiv E_{\tilde{x}_t} E_{x_t}$. Also, $\widetilde{Var}_t(x_t) = Var_{\tilde{x}_t}(x_t) = \Omega$ and $Var_t(x_t) = Var_{\hat{x}_t}(x_t) = \Omega$ and all $\epsilon$ terms have expectation zero under both $\tilde{E}_t$ and $E_t$ since the terms have expectation zero conditional on $x_t$.

Under these assumptions, the expression for the risk-free rate becomes

$$
\begin{aligned}
1 &= \beta R^f \Upsilon_t \tilde{E}_t \left[\exp\left(\log(\lambda_{t+1}/\lambda_t) + ((1-\gamma)\nu - 1)(\bar{g} + \rho\tilde{x}_t + \sigma_x\epsilon_{x,t+1} + \sigma_A\epsilon_{A,t+1})\right)\right] \\
&= \beta R^f \Upsilon_t \exp\left[\tilde{E}_t\left(\log(\lambda_{t+1}/\lambda_t)\right) + \frac{\widetilde{Var}_t\left(\log(\lambda_{t+1}/\lambda_t)\right)}{2} + ((1-\gamma)\nu - 1)(\bar{g} + \rho\tilde{x}_t)\right. \\
&\quad \left. + \frac{((1-\gamma)\nu-1)^2}{2}(\sigma_x^2 + \sigma_A^2) + \frac{((1-\gamma)\nu-1)^2\rho^2}{2}\widetilde{Var}_t(x_t)\right],
\end{aligned}
$$

which implies

$$
\begin{aligned}
r^f &= -\log(\beta) - \log(\Upsilon_t) - \left[\tilde{E}_t\left(\log(\lambda_{t+1}/\lambda_t)\right) + \frac{\widetilde{Var}_t\left(\log(\lambda_{t+1}/\lambda_t)\right)}{2} + (((1-\gamma)\nu) - 1)(\bar{g} + \rho\tilde{x}_t)\right. \\
&\quad \left. + \frac{((1-\gamma)\nu-1)^2}{2}(\sigma_x^2 + \sigma_A^2) + \frac{((1-\gamma)\nu-1)^2\rho^2}{2}\widetilde{Var}_t(x_t)\right],
\end{aligned}
$$

where $r^f = \log(R^f)$. Now

$$
\begin{aligned}
\log(\Upsilon_t) = \log \tilde{E}_t \left(\hat{J}(\hat{k}_{t+1}, \hat{x}_{t+1}) \exp(\nu(1-\gamma)g_{A,t+1})\right)^{-\alpha} \\
+ \frac{\alpha}{1-\alpha} \log \tilde{E}_t \left(\hat{J}(\hat{k}_{t+1}, \hat{x}_{t+1}) \exp(\nu(1-\gamma)g_{A,t+1})\right)^{1-\alpha} \\
= \quad -\alpha \tilde{E}_t \left(\log(\hat{J}(\hat{k}_{t+1}, \hat{x}_{t+1})) + (\nu(1-\gamma)g_{A,t+1})\right) \\
+ \alpha \tilde{E}_t \left(\log(\hat{J}(\hat{k}_{t+1}, \hat{x}_{t+1})) + (\nu(1-\gamma)g_{A,t+1})\right) + \text{variance terms},
\end{aligned}
$$

so that the direct effect of $\alpha$ on the risk-free interest rate cancels out as before. Based on these rules, we rewrite the risk-free rate as

$$
\begin{aligned}
r^f &= -\log(\beta) - \left[\tilde{E}_t\left(\log(\lambda_{t+1}/\lambda_t)\right) + \frac{\widetilde{Var}_t\left(\log(\lambda_{t+1}/\lambda_t)\right)}{2} + ((1-\gamma)\nu - 1)(\bar{g} + \rho\tilde{x}_t)\right. \\
&\quad \left. + \frac{((1-\gamma)\nu-1)^2}{2}(\sigma_x^2 + \sigma_A^2) + \frac{((1-\gamma)\nu-1)^2\rho^2}{2}\widetilde{Var}_t(x_t)\right] + \text{extra variance terms}
\end{aligned}
$$

To understand the impact of ambiguity aversion on the risk-free rate, we note that the log-linear rule for the risk-free interest implies that $r^f$ depends positively on the distorted posterior mean, $\tilde{x}_t$, and negatively on the distorted posterior variance, $\widetilde{Var}_t(x_t)$.[17] From the definition

[17]Notice that the term $((1-\gamma)\nu - 1)$ is negative for all values of $\gamma \geq 1$ since $0 < \nu < 1$.

of $\xi_t$, since the distorted posterior mean declines with increases in $\alpha$ due to the endogenous tilting of beliefs under ambiguity aversion while the distorted posterior variance increases with $\alpha$, we expect that an increase in ambiguity aversion will tend to reduce the risk-free rate. This may be interpreted as reflecting the increased demand for risk-free assets in an environment with endogenous doubt and pessimism. These effects on the risk-free rate are present in the endowment economy considered by Collard *et al.* (2012). In their case, a term similar to $\Upsilon_t$ is equal to unity, and the ratio of the Lagrange multipliers depends only on consumption growth, which they take as exogenous. In our case, the term $\lambda_{t+1}/\lambda_t$ depends on consumption and leisure choices, which are determined as functions of the transformed capital stock and the evolution of agents' beliefs.

## 5.2 Simulations

In Table 7, we report the average interest rate and the mean and standard deviation of "as if" or distorted beliefs that are implied by the smooth ambiguity model. We simulate 100 different economies of 275 observations each corresponding to the sample period 1947:I-2015:IV, and report values after discarding the burn-in sample of 1947:I-1978:IV. As in our other simulations exercises, we implement the simulations under the assumption that the TFP shocks are drawn from the high persistence/low variability versus low persistence/high variability processes. However, since the values of the average interest rate did not change significantly across the processes with high or low persistence, we only report the values for the high persistence process. We also consider the case of $\gamma = 2, \alpha = 5$ throughout the simulations.[18]

For future reference, we note that the mean and variance of the distorted or "as if" posterior distribution allow for uncertainty about the persistence parameter, and incorporate the probability that an agent places on the probability of the low versus the high persistence process are given as follows:

$$\tilde{x}_t = \eta_t \int_{-\infty}^{\infty} (x_{l,t}) \xi_t^{(l)} dF(x_{l,t}) dx_{l,t} + (1-\eta_t) \int_{-\infty}^{\infty} (x_{h,t}) \xi_t^{(h)} dF(x_{h,t}) dx_{h,t},$$

[18]It is also possible to use a higher value of $\beta = 0.9926$ (see, for example, Christiano and Eichenabum (1992)), which will tend to reduce the risk-free rate further but our focus here is on understanding the role of ambiguity.

and

$$\widetilde{Var}_t(x_t) = \eta_t \int_{-\infty}^{\infty} (x_{l,t}^2)\xi_t^{(l)} dF(x_{l,t})dx_{l,t} + (1-\eta_t)\int_{-\infty}^{\infty} (x_{h,t}^2)\xi_t^{(h)} dF(x_{h,t})dx_{h,t} - \tilde{x}_t^2.$$

In Table 7, we first report results for the baseline model with uninformative priors as well as the case with the Panel B estimates for the aggregate TFP process. These results show that a lower distorted mean and higher distorted variance tend to reduce the risk-free rate, as we argued above. However, one observation that we can make regarding the results for the baseline model as well as the Panel B estimates is that the variation in the distorted beliefs under these estimates is quite small. Hence, the notion of a "flight to safety" in response to the induced endogenous pessimism does not manifest itself in significantly low interest rates.

To obtain a better gauge of the role of ambiguity and ambiguity aversion, we conduct another counterfactual exercise by asking whether different measures of the underlying TFP process may provide additional insights. For this purpose, we re-estimate TFP growth processes using annual sectoral TFP measures provided by Bloom *et al.* (2012) for 4-digit manufacturing industries in the 2-digit SIC groups 24 Lumber and Wood Products, except Furniture, 28 Chemicals and Allied Products, 29 Petroleum Refining and Related Industries, 30 Rubber and Miscellaneous Products, 31 Leather and Leather Products, 33 Primary Metal Industries, 34 Fabricated Metal Products (except Machinery and Transportation Equipment), 35 Industrial and Commercial Machinery and Computer Equipment, 36 Electronic and Other Electrical Equipment and Components (except Computer Equipment), and 37 Transportation Equipment. As a gauge of the impact of uncertainty versus ambiguity, we also report the uncertainty measures computed by Bloom *et al.* (2012) for these industries.[19]

The results in Table 7 show that industries for which the distortions induced by ambiguity

[19] Bloom *et al.* (2012) use the Census of Manufacturers and the Annual Survey of Manufacturers to construct an establishment-level panel data set. To generate their TFP uncertainty measures, they define value-added TFP for each establishment $j$ in industry $i$ at time $t$ as $\log(z_{j,i,t} = \log(v_{j,i,t}) - \alpha_{i,t}^S \log(k_{j,i,t}^S) - \alpha_{j,i,t}^E \log(k_{j,i,t}^E) - \alpha_{i,t}^N \log(n_{j,i,t})$, where $v_{j,i,t}$ denotes value-added, $k_{j,i,t}^S$ is the stock of structures, $k_{j,i,t}^E$ is the stock of equipment, and $n_{j,i,t}$ is total hours worked, and $\alpha_{i,t}^k, k = S, E, N$ are the cost shares of the different inputs. TFP shocks $e_{j,i,t}$ are then defined as the residual from establishment-level log TFP from a first-order autoregressive equation with time and industry dummies. At the industry level, they then use interquartile range (IQR) and the weighted mean of absolute values as uncertainty measures. See their Appendix A for further descriptions.

and ambiguity aversion are relatively large (as shown by the distorted mean and standard deviation of $\tilde{x}_t$) also tend to display lower values of the risk-free rate. These industries include Petroleum Mining (2911), Plastic Pipes (3084), Steel Wiredrawing, Steel Nails and Spikes (3315), Internal Combustion Engines (3519), and Motors and Generators (3621). [20] Comparing the uncertainty measures used by Bloom *et al.* (2012) with our induced ambiguity measures, we find that for an industry such as Petroleum Mining (2911) which has a very high uncertainty measure, there is also evidence for the presence of significant ambiguity regarding the cyclical component of the underlying TFP process. This is evident in the low distorted mean and relatively high distorted standard deviation, which then yield a low value of the real interest rate of 1% per quarter. Given the volatile and unpredictable nature of technological developments in this industry such as fracking, one may rationalize the presence of high uncertainty and ambiguity for this sector. However, we observe that the uncertainty measures in Bloom *et al.* (2012) do not necessarily correlate with our ambiguity measures for all of the industries. As an example, Internal Combustion Engines (3519) has a high distorted standard deviation and a low implied risk-free rate equal to that for Petroleum Mining but its uncertainty measure is considerably lower compared to Petroleum Mining. On the other hand, an industry such as Shipbuilding and Repairing (3731) has an uncertainty index similar to that for Internal Combustion Engines but the distorted standard deviation for the cyclical component of TFP growth is relatively small, implying a relatively high risk-free rate.

To examine the relationship between the uncertainty index and our implied measures of ambiguity, we consider four additional industries which have some of the highest uncertainty indices, namely, Electronic Computers (3571), Soybean Oil Mills (2075) and Pesticides and Agricultural Chemicals, Not Elsewhere Classified (2879) and one industry with one of the lowest

[20]In this section, we do not report the business cycle moments implied under the sectoral TFP estimates, though they are available upon request. Nevertheless, we choose industries for which the relative volatilities of output, consumption and investment are comparable to the aggregate TFP estimates, although the overall volatilities may differ across different industries. Another issue which we have not addressed when using the sectoral TFP estimates in the aggregate business cycle model is the existence of increasing returns to scale in production at the sectoral level, which would lead to model mis-specification. Hall (1988, 1990) argued for the existence of increasing returns and significant market power in many US industries. Subsequently, however, authors such as Basu and Fernald (1997) and Burnside (1996) suggest that US manufacturing industry essentially exhibits constant returns to scale.

uncertainty indices, Glass Containers (3221). We find that the distorted standard deviation for Pesticides and Agricultural Chemicals is the highest among all of the industries that we consider, and this translates into a relatively low value for the risk-free rate. Thus, we find that the information conveyed by the uncertainty and ambiguity measures corroborate each other for this industry. This is also the case for Petroleum Mining, which has a low distorted mean and a high uncertainty measure, as we argued earlier. However, Electronic Computers possesses a high distorted variance but also it also has distorted cyclical mean that is very large, implying a high value for the risk-rate. Evidently, the uncertainty measures reported by Bloom *et al.* (2012) and the ambiguity measures that we report can diverge for key industries. Conversely, for a low uncertainty industry such as Glass Containers, a relatively small distorted mean and a relatively low distorted standard deviation lead to a moderately low interest rate.

To understand the reasons for these findings, we display the filtered means $\hat{x}_{lt}$ and $\hat{x}_{ht}$ for the different processes in Figure 8. We observe that for the baseline model based on aggregate TFP, the values of both $\hat{x}_{lt}$ and $\hat{x}_{ht}$ are very small. Moreover, there is very little variation in either of these measures, implying very low variation for the distorted variable, $\tilde{x}_t$, as well. When we allow for a more informative prior on the low persistence process, as in the Panel B model, the filtered means display more variability, implying that the distorted variance also tends to increase compared to the baseline model. When we turn to the sectoral results, the variability of the filtered means increases significantly, which, in turn, magnifies the distorted variances. Another way of understanding the impact of ambiguity is to note that an ambiguity-averse agent endogenously behaves as if the uncertainty is more persistent and severe following negative shocks than in normal times. From Figure 8, we observe that for an industry such as Petroleum Mining, the mean of the high persistence process tends display significant drops, and to fall below the mean of the low persistence process. Since ambiguity-averse agents forecast TFP growth by putting more weight on the "worst case persistence", in situations with negative shocks where $\hat{x}_{ht} < \hat{x}_{lt}$, the worst case persistence is $\rho_h$, suggesting that the economy will remain in the bad state for a long time. This tends to increase the endogenous distortions and to lead to greater ambiguity compared to situations for which $\hat{x}_{ht}$ does not fall systematically below $\hat{x}_{lt}$.

Figure 8: Filtered beliefs underlying the risk-free rate values - Aggregate TFP and Industry Results

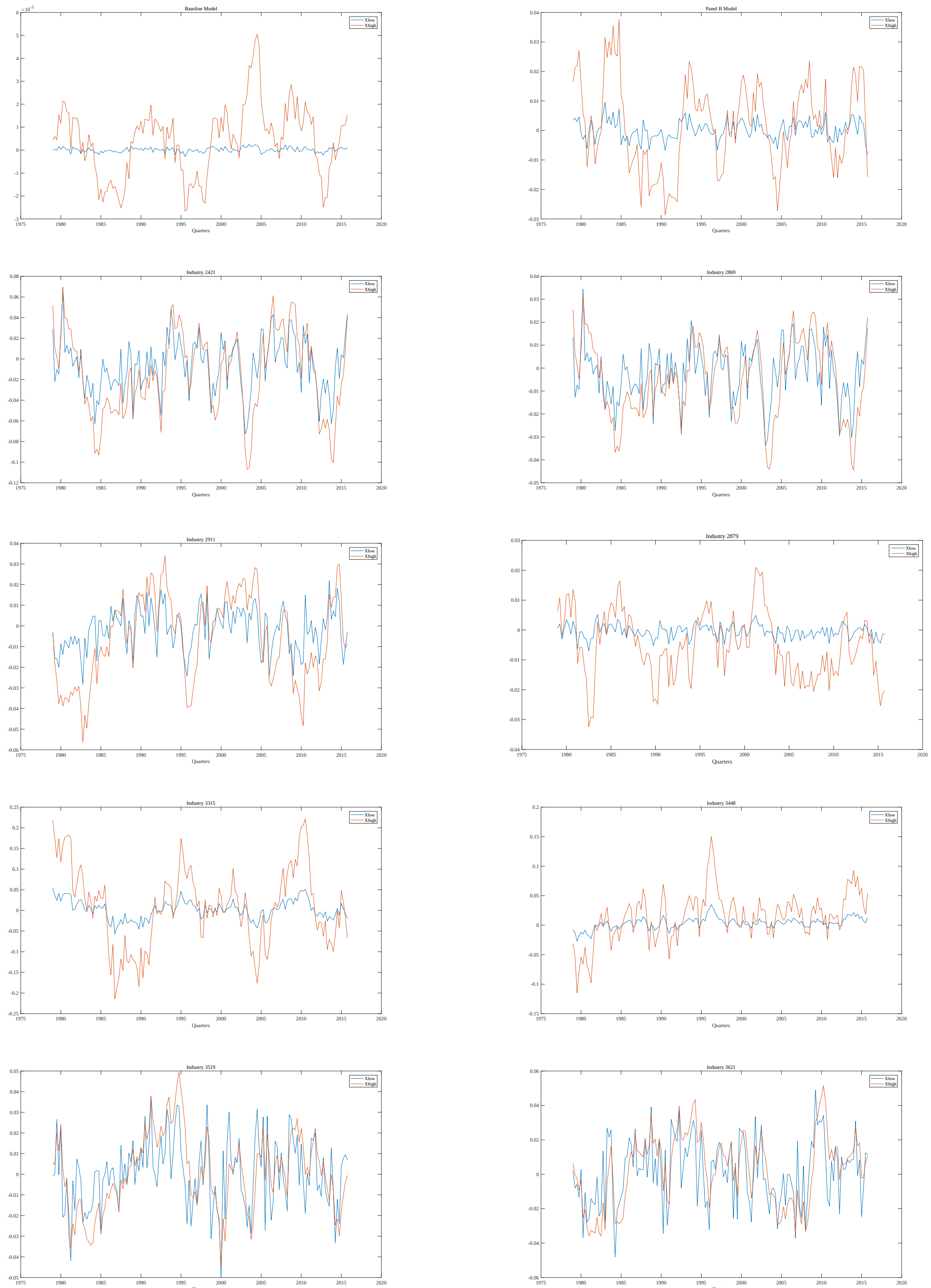
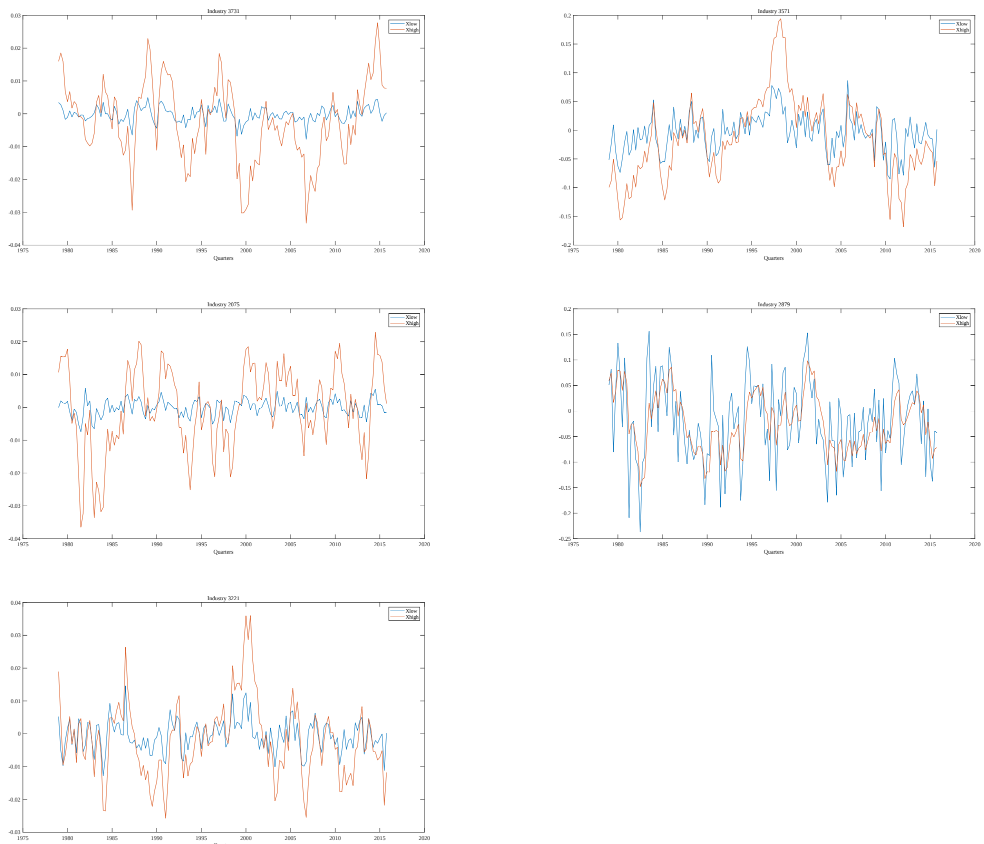


Figure 9: Filtered beliefs underlying the risk-free rate values - Aggregate TFP and Industry Results (cont'd)



By contrast, for Electronic Computers, we observe the converse, namely, that the mean of the high persistence process, $\hat{x}_{ht}$, rose above that of the low persistence process, $\hat{x}_{lt}$, strongly during the dotcom bubble in the early 2000's. When $\hat{x}_{ht} > \hat{x}_{lt}$, the worst case persistence is $\rho_l$, which suggests that the good state is relatively transient. Given the very large deviation displayed by the cyclical mean during such a good state, the distortion tends to be relatively minor for this industry, despite its very large uncertainty measure. This leads to a higher risk-free rate, which is consistent with the underlying theory.

# 6 Conclusion

In this paper, we have examined the cyclical dynamics of a Real Business Cycle model with ambiguity averse consumers and investment irreversibility using the smooth ambiguity model of Klibanoff *et al.* (2005, 2009). In this model, agents do not know which distribution the unobserved temporary component of TFP growth is coming from, and learn about it based on observations of current and past values of the model's variables. The existence of such ambiguity combined with ambiguity aversion on the part of agents may generate the cyclical dynamics of real variables such as consumption, investment and output) through the asymmetric effects of uncertainty aversion in booms versus recessions. Surprisingly, our results imply that the intertemporal substitution in consumption and leisure combined a real investment friction such as irreversible investment operating through the transmission channels of the standard Real Business Cycle model dominate the impact of uncertainty aversion when agents can choose to optimally smooth consumption through investment and hours worked choices in response to labor-augmenting technology shocks. However, ambiguity and ambiguity aversion affect endogenous choices through information and learning effects.

We interpret these results as implying that, on average, the behavior of quantities may be well approximated by the optimal response of consumption, investment, and hours worked to exogenous changes in productivity, as Kydland and Prescott (1982) originally argued. As the history of Real Business Cycle analysis has shown, modifications of the original model of cyclical fluctuations proposed by Kydland and Prescott (1982) have allowed us to understand other transmission mechanisms that were not included in the original model of aggregate fluctuations. What we have shown here is that the addition of ambiguity and ambiguity aversion in a theoretically well grounded and empirically consistent manner generates precautionary saving motives in response to uncertainty shocks but the main mechanisms of the Real Business Cycle involving intertemporal substitution effects remain. Nevertheless, our results have revealed a role for learning and information even in the highly aggregative environment considered, suggesting ambiguity averse agents' actions will be affected by the flow of information about fundamentals across the business cycle.

In his original analysis, Tallarini (2000) showed that in the type of environments with full risk sharing that we have considered here, uncertainty aversion will tend to manifest itself in asset prices. Here we have examined the behavior of the risk-free rate, and shown that the endogenous distortion in beliefs arising from the behavior of the temporary component of TFP growth will tend to amplify the behavior of asset prices. In our analysis, we have not considered the behavior of the risky rate of return nor quantities such as the equity premium. However, Collard *et al.* (2012) note that "ambiguity aversion gets the first moment of equity premium right by holding down the risk-free rate while affecting the risky rate only very marginally." Tallarini (2000) notes that generating an equity premium that is consistent with the data in a production economy requires the addition of frictions of adjustment costs or other frictions that will generate variation in the price of capital. While we have considered the presence of a friction such as investment irreversibility, our focus is not directly on accounting for various asset pricing phenomena considered in the literature.

Undoubtedly, uncertainty and ambiguity aversion are important factors deriving agents' decisions to work, to invest, and to consume. Our results suggest that relaxing the assumption of complete markets/perfect insurance and introducing various forms of market incompleteness may work towards generating a greater role for uncertainty and ambiguity aversion. Second, an environment that considers individual heterogeneity may be more likely to capture the impact of such uncertainty aversion. A third issue has to do with the measurement of TFP shocks in order to gauge their influence on business cycles. As we have shown, the highly aggregative nature of the TFP process considered here does not lead to significant sources of uncertainty and ambiguity. Hence, using more disaggregated measures of productivity changes at the firm or industry level in a general equilibrium economy with multiple sectors may provide more useful approaches to generating the impact of uncertainty and ambiguity aversion. We leave exploring these avenues for future work.

# A The stationarity inducing transformations

In the following discussion, we consider generating stationarity-inducing transformations under alternative parameterizations for KMM smooth ambiguity preferences. These include the power-power and log-exponential pairings for the current utility and ambiguity functions, $u(\cdot)$ and $\phi(\cdot)$, respectively. These specifications are similar to the cases considered by Ju and Miao (2007). We make use of the generic social planner's problem described of Section 2.3 of the paper.

**Case (i)**

In this case, $u(c) = (c^{\nu} l^{1-\nu})^{1-\gamma}/(1-\gamma), \gamma \geq 0,\ 0 \leq \nu \leq 1$ and $\phi(x) = x^{1-\alpha}/(1-\alpha), \alpha \geq 0$. Using $y = \phi(x)$, we can show that

$$\phi^{-1}(y) = [(1-\alpha)y]^{\frac{1}{1-\alpha}}.$$

Substitute for $u(\cdot), \phi(\cdot), \phi^{-1}(\cdot)$ into the generic representation for the indirect value function $J$ as follows.

$$J(k_t, \mu_t, A_t) = \max_{c_t, i_t, n_t} \left\{ \frac{(c_t^{\nu} l_t^{1-\nu})^{1-\gamma}}{1-\gamma} + \frac{\beta}{1-\gamma} \left[ (1-\alpha) E_{\mu_t} \left( \frac{[E_{x_t}(1-\gamma) J(k_{t+1}, \mu_{t+1}, A_{t+1})]^{1-\alpha}}{1-\alpha} \right) \right]^{\frac{1}{1-\alpha}} \right\}.$$

We argue that the stationarity inducing transformation is defined in terms of the transformed variables of the model as

$$\left\{ \hat{c}_t, \hat{i}_t, \hat{k}_t, \hat{y}_t \right\} = \left\{ \frac{c_t}{A_t}, \frac{i_t}{A_t}, \frac{k_t}{A_t} \right\}.$$

Using the homogeneity of the indirect value function, we have that

$$J(k_t, \mu_t, A_t) = \hat{J}(\hat{k}_t, \mu_t) A_t^{\nu(1-\gamma)}. \tag{A.1}$$

Using this relationship, dividing both sides of the indirect function by $A_t^{1-\gamma}$ and the constraints by $A_t$, we obtain

$$\hat{J}(\hat{k}_t, \mu_t) = \max_{c_t, i_t, n_t} \left\{ \frac{(c_t^{\nu} l_t^{1-\nu})^{1-\gamma}}{(1-\gamma) A_t^{\nu(1-\gamma)}} + \frac{\beta}{1-\gamma} \left[ E_{\mu_t} \left( \left[ \frac{E_{x_t}(1-\gamma)\hat{J}(\hat{k}_{t+1}, \mu_{t+1}) A_{t+1}^{\nu(1-\gamma)}}{A_t^{\nu(1-\gamma)}} \right]^{1-\alpha} \right) \right]^{\frac{1}{1-\alpha}} \right\}$$

subject to

$$\frac{c_t}{A_t} + \frac{i_t}{A_t} \leq \frac{k_t^a (A_t n_t)^{1-a}}{A_t},$$

$$\frac{k_{t+1}}{A_t} = (1-\delta)\frac{k_t}{A_t} + \frac{i_t}{A_t},$$

$$\frac{i_t}{A_t} \geq 0,$$

Hence, the transformed value function satisfies

$$\hat{J}(\hat{k}_{t+1}, \mu_t) = \max_{\hat{c}_t, \hat{i}_t, n_t} \left\{ \frac{(\hat{c}_t^{\nu} l_t^{1-\nu})^{1-\gamma}}{1-\gamma} + \right.$$

$$\left. \frac{\beta}{1-\gamma} \left[ E_{\mu_t} \left( \left[ E_{x_t} \left( (1-\gamma)\hat{J}(\hat{k}_{t+1}, \mu_{t+1}) \right) (\exp(\nu(1-\gamma) g_{A,t+1}) \right]^{1-\alpha} \right) \right]^{\frac{1}{1-\alpha}} \right\} \tag{A.2}$$

subject to the constraints

$$\hat{c}_t + \hat{i}_t \leq \hat{k}_t^a,$$

$$\exp(g_{A,t+1})\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \hat{i}_t,$$

$$\hat{i}_t \geq 0.$$

Notice that the assumed form of the indirect value function yields a well-defined representation of the power-power variety of smooth ambiguity preferences with $\alpha > \gamma$ and $\gamma > 0$. When $\gamma < 1$, the affine transformation by $1-\gamma$ does not matter. By contrast, when $\gamma > 1$, the representation involves taking a power of a positive function, $(1-\gamma)\hat{J}(\hat{k}_t, \mu_t)$.

**Case (ii)**

In this case, $u(c) = \ln(c)$ and $\phi(x) = -\exp(-\alpha x)/\alpha$. Again substitute for $u(\cdot), \phi(\cdot), \phi^{-1}(\cdot)$ into the generic representation for the indirect value function $J$ as follows.

$$\begin{aligned} \phi^{-1}(x) &= -\frac{1}{\alpha}\ln(-\alpha y) \\ J(k_t, \mu_t, A_t) &= \max_{c_t, i_t, n_t}\left\{\ln(c_t) + \ln(l_t) - \frac{\beta}{\alpha}\ln\left[E_{\mu_t}\left(\exp(-\alpha E_{x_t} J(k_{t+1}, \mu_{t+1}, A_{t+1}))\right)\right]\right\} \end{aligned}$$

We argue that the following transformation will be stationarity inducing:

$$\left\{\hat{c}_t, \hat{i}_t, \hat{y}_t, \hat{k}_t\right\} = \left\{\frac{c_t}{A_t}, \frac{i_t}{A_t}, \frac{y_t}{A_t}, \frac{k_t}{A_t}\right\},$$

where

$$J(k_t, \mu_t, A_t) = \hat{J}_t(\hat{k}_t, \mu_t) + \frac{\ln(A_t)}{1-\beta}.$$

Making these substitutions for $J_t$ and $\hat{J}_t$ yields

$$\begin{aligned} \hat{J}(\hat{k}_t, \mu_t) &= \max_{c_t, i_t, n_t}\left\{\ln(c_t) + \ln(l_t) - \frac{\ln(A_t)}{1-\beta} - \frac{\beta}{\alpha}\ln\left[E_{\mu_t}\left(\exp\left(-\alpha E_{x_t} J(k_{t+1}, \mu_{t+1})\right)\right)\right]\right\} \\ &= \max_{c_t, i_t, n_t}\left\{\ln(\hat{c}_t) + \ln(l_t) - \frac{\beta}{\alpha}\left(\ln\left[E_{\mu_t}\left(\exp\left(-\alpha E_{x_t}\left(\hat{J}(\hat{k}_{t+1}, \mu_{t+1}) + \frac{\ln(A_{t+1})}{1-\beta}\right)\right)\right)\right] + \frac{\alpha\ln(A_t)}{1-\beta}\right)\right\} \\ &= \max_{c_t, i_t, n_t}\left\{\ln(\hat{c}_t) + \ln(l_t) - \frac{\beta}{\alpha}\ln\left[E_{\mu_t}\left(\exp\left(-\alpha E_{x_t}\left(\hat{J}(\hat{k}_{t+1}, \mu_{t+1}) + \frac{\ln(A_{t+1})}{1-\beta}\right)\cdot A_t^{\frac{\alpha}{1-\beta}}\right)\right)\right]\right\}. \end{aligned}$$

Therefore, the transformed problem using the guess function for $\hat{J}(\hat{k}_t, \mu_t)$ that we specified above is expressed as

$$\hat{J}(\hat{k}_t), \mu_t) = \max_{c_t, i_t, n_t}\left\{\ln(\hat{c}_t) + \ln(l_t) - \frac{\beta}{\alpha}\ln\left[E_{\mu_t}\left(\exp\left(-\alpha E_{x_t}\left(\hat{J}(\hat{k}_{t+1}, \mu_{t+1}) + \frac{g_{A,t+1}}{1-\beta}\right)\right)\right)\right]\right\} \quad \text{(A.3)}$$

subject to the constraints defined above.

# B Numerical solution approach

We now describe how to numerically solve the social planner's problem described in Section 2.3. Our task is we determine the function $\hat{J}(\hat{k}_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t)$ for all values of the normalized capital stock, $\hat{k}_t$, and for the variables specifying beliefs, $\hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t)$. Unlike Collard *et al.* (2016) who consider an endowment economy, we also need to calculate the optimal investment policy as part of the numerical solution for the indirect value function. Notice that the optimization routine needs to account for the inequality constraint on the choice of $\hat{i}_t$.[21] We use the method of value iteration with Chebyshev interpolation, which involves approximating the function $\hat{J}(\hat{k}_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t)$ by a parametric function whose coefficients are determined according to a minimum residual method; see Judd (1998)

We begin by explicitly writing the expectations that appear in this formulation.

$$\hat{J}(\hat{k}_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) = \max_{\hat{c}_t, n_t, \hat{i}_t} \Bigg\{ \frac{((\hat{c}_t{}^{\nu} l_t^{1-\nu})^{1-\gamma}}{1-\gamma} +$$

$$\beta \Bigg[ \eta_t \left( \int_{-\infty}^{\infty} \left( \int\int_{-\infty}^{\infty} \hat{J}(\hat{k}^{(l)}_{t+1}, \hat{x}^{(l)}_{h,t+1}, \hat{x}^{(l)}_{l,t+1}, \eta^{(l)}_{t+1}) \exp(\nu(1-\gamma) g_{A_l,t+1}) dF(\varepsilon_{l,t+1}) \right)^{1-\alpha} dF(x_{l,t}) \right) +$$

$$(1-\eta_t) \left( \int_{-\infty}^{\infty} \left( \int\int_{-\infty}^{\infty} \hat{J}(\hat{k}^{(h)}_{t+1}, \hat{x}^{(h)}_{h,t+1}, \hat{x}^{(h)}_{l,t+1}, \eta^{(h)}_{t+1}) \exp(\nu(1-\gamma) g_{A_h,t+1}) dF(\varepsilon_{h,t+1}) \right)^{1-\alpha} dF(x_{h,t}) \right) \Bigg]^{\frac{1}{1-\alpha}} \Bigg\}.$$

subject to

$$\hat{c}_t + \hat{i}_t \leq \hat{k}_t^a n_t^{1-a},$$

$$\exp(g_{A,t+1}) \hat{k}_{t+1} = (1-\delta)\hat{k}_t + \hat{i}_t,$$

$$l_t + n_t \leq 1,$$

$$\hat{i}_t \geq 0.$$

Here $\hat{x}^{(k)}_{h,t+1}, \hat{x}^{(k)}_{l,t+1}, \eta^{(k)}_{t+1}$ are functions of $\varepsilon_{k,t+1} = (\epsilon_{x_k,t+1}, \epsilon_{A_k,t+1})'$, $k = l, h$, which is a 2 by 1 vector standard normal shocks and $\eta^{(l)}_{t+1}$ is the posterior probability at time $t+1$ that the model with

[21] For details of the solution procedure, see Ju and Miao (2012), Jahan-Parvar and Liu (2012), Collard *et al.* (2016), and Liu and Zhang (2015), among others.

$\rho_l$ is the true data generating process. Notice that next period's capital stock is also indexed by the different stochastic processes for the technology shock next period that is assumed to be the true process. $F(\varepsilon_{k,t+1}), k = l, h$ are both bivariate independent standard normal distributions while $F(x_{k,t}), k = l, h$ is a normal distribution with mean $\hat{x}_{k,t}$ and variance $\Omega_k$, which is defined below.

## B.1 Updating beliefs

The updates for $\hat{x}^{(i)}_{k,t+1}$ are obtained using the Kalman filter algorithm as follows:

$$\begin{aligned}
\hat{x}^{(l)}_{l,t+1}(\varepsilon_{l,t+1}) &= \rho_l \hat{x}_{l,t} + K_l \nu^{(l)}_{l,t+1}, \\
\hat{x}^{(l)}_{h,t+1}(\varepsilon_{l,t+1}) &= \rho_h \hat{x}_{h,t} + K_h \nu^{(l)}_{h,t+1}, \\
\hat{x}^{(h)}_{l,t+1}(\varepsilon_{h,t+1}) &= \rho_l \hat{x}_{l,t} + K_l \nu^{(h)}_{l,t+1}, \\
\hat{x}^{(h)}_{h,t+1}(\varepsilon_{h,t+1}) &= \rho_h \hat{x}_{h,t} + K_h \nu^{(h)}_{h,t+1},
\end{aligned}$$

where $\nu^{(i)}_{k,t+1}, (i) = l, h, k = l, h$ are the "surprises". For example, when the DGP is $(i) = l$ and the filter uses $\rho_k, k = h$, the surprise is defined as

$$\begin{aligned}
\nu^{(l)}_{h,t+1} &= g_{A_l,t+1} - \bar{g} - \rho_h \hat{x}_{h,t} \\
&= \bar{g} - \bar{g} + \rho_l x_{l,t} - \rho_h \hat{x}_{h,t} + \sigma_{x_l}\epsilon_{x_l,t+1} + \sigma_{A_l}\epsilon_{A_l,t+1} \\
&= \rho_l x_{l,t} - \rho_h \hat{x}_{h,t} + \sigma_{x_l}\epsilon_{x_l,t+1} + \sigma_{A_l}\epsilon_{A_l,t+1}.
\end{aligned}$$

The Kalman gain parameters, $K_k, k = l, h$, depending on whether the low or high persistence model is assumed to be the true model, respectively, are

$$K_k = \rho_k \Omega_k f_k^{-1}, \quad f_k = \Omega_k + \sigma^2_{A_k},$$

where $f_k = E[(g_{A_k,t+1} - E(g_{A_k,t+1}))^2 | g_{A,1}, \ldots, g_{A,t}]$, and $\Omega_k = E[(x_{k,t+1} - \hat{x}_{k,t+1})^2 | g_{A,1}, \ldots, g_{A,t}]$, $k = l, h$ are defined as the solution to

$$\Omega_k = \rho_k^2 \Omega_k - \rho_k^2 \Omega_k^2 f_k^{-1} + \sigma^2_{x_k}.$$[22]

The Bayes update of $\eta_t$ is obtained as follows:

$$\begin{aligned}
\eta_{t+1}^{(l)}(\varepsilon_{l,t+1}) &= \frac{\eta_t L(\nu_{l,t+1}^{(l)}, f_l)}{\eta_t L(\nu_{l,t+1}^{(l)}, f_l) + (1-\eta_t) L(\nu_{h,t+1}^{(l)}, f_h)}, \\
\eta_{t+1}^{(h)}(\varepsilon_{h,t+1}) &= \frac{\eta_t L(\nu_{l,t+1}^{(h)}, f_l)}{\eta_t L(\nu_{l,t+1}^{(h)}, f_l) + (1-\eta_t) L(\nu_{h,t+1}^{(h)}, f_h)},
\end{aligned}$$

where the likelihood is

$$L(\nu_{j,t+1}^{(i)}, f_j) = \frac{1}{2\pi\sqrt{f_j}} \exp\left(-\frac{(\nu_{j,t+1}^{(i)})^2}{2f_j}\right).$$

## B.2 The numerical algorithm

Following the recent literature (see, e.g. Walker *et al.* (2014), we approximate the indirect value function $\hat{J}(\hat{k}_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t)$ by a parametric function of the form

$$\Phi(X_t) = \exp\left(\sum_{i_k, i_l, i_h, i_\eta \in \Upsilon} c_{i_k, i_l, i_h, i_\eta} T_{i_k}(\hat{k}_t) T_{i_l}(\hat{x}_{l,t}) T_{i_h}(\hat{x}_{h,t}) T_{i_\eta}(\hat{\eta}_t)\right), \tag{B.1}$$

where $X_t = (\hat{k}_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \hat{\eta}_t)$ denotes the vector of state variables. Notice that in the full information case, the vector of state variables is given by $X_t = (\hat{k}_t, x_t)$ where $x_t$ is temporary component of the technology shock. In this expression, the set of indices is defined as $\Upsilon = \{i_z = 1, \ldots, n_z; z \in \{k, l, h, \eta\} | i_k + i_l + i_h + i_\eta \leq \max(n_k, n_l, n_h, n_\eta)\}$. This definition assumes that we are considering a complete basis of polynomials. In this expression $T_n(\cdot)$ is the Chebyshev

[22] These results are obtained by applying the Kalman filter algorithm to the state and measurement equations as

$$\begin{aligned}
x_{k,t+1} &= \rho_k x_{k,t} + \sigma_{x_k} \epsilon_{x_k,t+1}, \quad k = l, h, \\
g_{A_k,t+1} - \bar{g} &= x_{k,t+1} + \sigma_{A_k} \epsilon_{A_k,t+1}, \quad k = l, h.
\end{aligned}$$

The expression for the gain parameters $K_k$ and the variances of the filtered estimates of $x_{k,t}$ denoted $\Omega_k$ are obtained as a direct application of the Kalman filter algorithm. A similar application of the Kalman filter yields the expressions for $x_{k,t+1}^{(i)}$ and the surprises $\nu_{k,t+1}^{(i)}$ for $k = l, h$ and $i = l, h$. See Anderson and Moore (1979), Ch. 3.

functions $T(\cdot)$ are defined as the unique polynomials satisfying

$$T_n(x) = \cos(n \arccos x),$$

or equivalently, $T_n(\cos(\nu)) = \cos(n\nu)$. Using a recursive formulation, we have $T_0(x) = 1, T_1(x) = x$ and $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$. Using the definition that $\cos\left((2k+1)\frac{\pi}{2}\right) = 0$, it is possible to show that the roots of $T_n$ are

$$x_k = \cos\left(\frac{2k-1}{2n}\pi\right), \quad k = 1, \ldots, n.$$

The roots of the Chebyshev polynomial are also called Chebyshev nodes because they are used as nodes in polynomial interpolation.[23] The orders of the Chebyshev functions $T_{i_z,n+1}(x)$ are set as $(n_k, n_{x_h}, n_{x_l}, n_\eta) = (4, 2, 2, 2)$, and 8 nodes each are used to evaluate the Chebyshev functions, yielding a total of 4096 nodes

Next, define the functions

$$\begin{aligned} \kappa_l(x_{l,t}) &= \left(E_{x_{l,t}} \Phi(X_{t+1}^{(l)}) \exp((1-\gamma) g_{A_l,t+1})\right)^{1-\alpha} \\ &= \left(\int_{\infty}^{\infty} \int_{\infty}^{\infty} \Phi(X_{t+1}^{(l)}) \exp((1-\gamma) g_{A_l,t+1}) dF(\varepsilon_{l,t+1})\right)^{1-\alpha} \end{aligned} \tag{B.2}$$

and

$$\begin{aligned} \kappa_h(x_{h,t}) &= \left(E_{x_{l,t}} \Phi(X_{t+1}^{(h)}) \exp((1-\gamma) g_{A_h,t+1})\right)^{1-\alpha} \\ &= \left(\int_{\infty}^{\infty} \int_{\infty}^{\infty} \Phi(X_{t+1}^{(h)}) \exp((1-\gamma) g_{A_h,t+1}) dF(\varepsilon_{h,t+1})\right)^{1-\alpha} \end{aligned} \tag{B.3}$$

Notice that the indirect value function can be expressed as

$$\hat{J}(\hat{k}_t, \hat{x}_{l,t}, \hat{x}_{h,t}, \eta_t) = \max_{\hat{i}_t} \left\{ \frac{(\hat{k}_t^a - \hat{i}_t)^{1-\gamma}}{1-\gamma} + \left[\eta_t E_{\hat{x}_{l,t}} \kappa_l(x_{l,t}) + (1-\eta_t) E_{\hat{x}_{h,t}} \kappa_h(x_{h,t})\right]^{\frac{1}{1-\alpha}} \right\}.$$

To be able to evaluate the value function, we also need to approximate the integrals that appear in this expression using numerical integration procedures.

[23] This approach is followed by Ju and Miao (2012) and Jahan-Parvar and Liu (2012) in their asset pricing applications with Markov switching processes.

- Gauss-Hermite quadrature: In the case of uni-dimensional integrals (as in the outer integral involved in the computation of expectations such as (B.4)), a Gauss Hermitian quadrature approach. Specifically, consider the expectations of the form $E_{\hat{x}_{k,t}}[K_k(x_{k,t})], k = l, h$. Since $x_{k,t}$ is distributed as normal with mean $\hat{x}_{k,t}$ and variance $\Omega_k$, we apply a change of variables $z_{k,t} = (x_{k,t} - \hat{x}_{k,t})/\sqrt{2\Omega_k}$ to write the Guass-Hermite quadrature rule as

$$\begin{aligned} E_{\hat{x}_{k,t}}[\kappa_k(x_{k,t})] &= \pi^{-1/2} \int_{-\infty}^{\infty} \kappa_k(\sqrt{2\Omega_k}(z_{l,t} + \hat{x}_{k,t}))dF(x_{k,t}), \\ &\approx \pi^{-1/2} \sum_{i=1}^{n} \omega_i \kappa_k(\sqrt{2\Omega_k}(z_{k,t} + \hat{x}_{k,t})) \end{aligned}$$

where $\omega_i = 2^{n+1} n! \sqrt{\pi}[H_{n+1}(x_i)]^{-2}$ and $H_{n+1}$ is the Hermite polynomial of order $n$.

- The monomial approach: In the case, of multi-dimensional integrals such as (B.2) or (B.3), Collard *et al.* (2016) use a monomial approach; see Judd (1998), p. 271-276) with 5 degree rule for an integrand on an unbounded range weighted by the standard normal. Specifically, we approximate the multi-dimensional integral

$$\int_{\infty}^{\infty} \int_{\infty}^{\infty} \Phi(X_{t+1}^{(k)}) \exp((1-\gamma) g_{A_k,t+1})) dF(\varepsilon_{k,t+1})$$

by a 5 degree rule using $2d+1$ points with $d = 2$ as

$$a_0 \Phi(0) + a_1 \sum_{i=1}^{d} (\Phi(re^i) + \Phi(-re^i)) + a_2 \sum_{i=1}^{d-1} \sum_{j=1}^{d} \left( \Phi(\pm se^i + \pm se^j) \right), \tag{B.4}$$

where $e^i$ denotes the $i$th column vector of the identity matrix of order $d = 2$, and

$$\begin{aligned} &r = \sqrt{1 + \frac{1}{2}d}, \quad s = \sqrt{\frac{1}{2} + \frac{d}{4}}, \quad v = \pi^{d/2}, \\ &a_0 = \frac{2}{d+2} v, \quad a_1 = \frac{4-d}{2(d+2)^2}, \quad a_2 = \frac{v}{(d+2)^2}. \end{aligned}$$

Suppose we obtain an approximation to the indirect value function at the $\tau$'th iteration using these steps. This will be based on the Chebyshev coefficients at the $\tau$'th stage of the algorithm, $\mathbf{c}^\tau$. Denote the approximation obtained by using these coefficients by $\hat{J}^{(\tau)}(X_t)$. Also define the vector of future state variables by $X_{t+1}^{(k)} = (\hat{k}_{t+1}^{(k)}, \hat{x}_{h,t+1}^{(k)}, \hat{x}_{l,t+1}^{(k)}, \eta_{t+1}^{(l)}), k = l, h$. The value function and optimal investment policy functions, $\hat{J}^*(X_t)$ and $\hat{i}_t^* = g(X_t)$, are obtained as the solution to

$$\hat{J}^*(X_t) = \max_{\hat{i}_t} \left\{ \frac{((\hat{k}_t^a n_t^{1-a} - \hat{i}_t)^\nu n_t^{1-\nu})^{1-\gamma}}{1-\gamma} + \beta \left[ \eta_t E_{\hat{x}_{l,t}} \left( E_{x_{l,t}} J^{(\tau)}(X_{t+1}^{(l)}) \exp(\nu(1-\gamma) g_{A_l,t+1}) \right)^{1-\alpha} \right.\right.$$
$$\left.\left. +(1-\eta_t) E_{\hat{x}_{h,t}} \left( E_{x_{h,t}} J^{(\tau)}(X_{t+1}^{(h)}) \exp(\nu(1-\gamma) g_{A_h,t+1}) \right)^{1-\alpha} \right]^{\frac{1}{1-\alpha}} \right\}.$$

subject to

$$\exp(g_{A,t+1})\hat{k}_{t+1} = (1-\delta)\hat{k}_t + \hat{i}_t,$$
$$\hat{i}_t \geq 0.$$

Now, at the end of the $\tau$'th iteration, the new value function $\hat{J}^*(X_t)$ is used to update the coefficients of the Chebyshev polynomials, and to obtain $\hat{J}^{(\tau+1)}(X_t)$ as we describe below.

Denote by $\mathbf{c}^\tau, \mathbf{c}^{\tau+1}$ as the set of coefficients entering (B.1) at the $\tau$'th and $\tau+1$'th stages, respectively. We determine the set of coefficients, $\mathbf{c}^{\tau+1}$ at the $\tau+1$'th stage, using a minimum weighted residual method. Recall that the indirect value function depends on the coefficients from the $\tau$'th stage as $J^*(X_t; \mathbf{c}^\tau)$ while the new approximation for the indirect value function depends on $\mathbf{c}^{\tau+1}$. The residual function associated with the new set of Chebyshev coefficients is given by $\mathcal{R}(X_t; \mathbf{c}^{\tau+1})$, where

$$\mathcal{R}(X_t; \mathbf{c}^{\tau+1}) = \Phi(X_t; \mathbf{c}^{\tau+1}) - \hat{J}^*(X_t; \mathbf{c}^\tau).$$

This involves solving the problem

$$\min_{\mathbf{c}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\Phi(X_t; \mathbf{c}^{\tau+1}) - \hat{J}^*(X_t; \mathbf{c}^\tau))^2 \omega(X_t) dX_t, ,$$

where $\omega(X)$ is a multi-dimensional weighting function. The first-order conditions for this problem with respect to elements of $c_{i_z}, i_z \in \Upsilon$ are

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\Phi(X_t;\mathbf{c}^{\tau+1})-\hat{J}^*(X_t;\mathbf{c}^{\tau}))T_{i_k}(\hat{k}_t)T_{i_l}(\hat{x}_{l,t})T_{i_h}(\hat{x}_{h,t})T_{i_\eta}(\eta_t)\omega(X_t)dX_t = 0. \quad \text{(B.5)}$$

If we assume that the weighting function is the product of the weights for $z \in \{k, l, h, \eta\}$ defined as

$$\omega_{i_z}(y_z) = \frac{T_{i_z}(y_z)}{\sqrt{1-y_z^2}}, \quad i_z = 1, \ldots, n_z, i_z \in \Upsilon,$$

where $y \in \{\hat{k}, \hat{x}_l, \hat{x}_h, \eta\})$, the integral in the orthogonality conditions can be solved using Gauss-Chebyshev quadrature. For integrals of this form, the quadrature nodes and the (constant) quadrature weights are given by

$$y_{j_z} = \cos\left(\frac{2j_z - 1}{2n_z}\pi\right) \text{ and } \omega_{j_z} = \pi/n_z, \quad j_z = 1, \ldots, m_z.$$

Hence, the integral in (B.5) is written as

$$\sum_{j_k, j_i, j_h, j_\eta} \mathcal{R}(y_{j_k}, y_{j_l}, y_{j_h}, y_{j_\eta}; \mathbf{c}^{\tau+1})T_{i_k}(y_{j_k}T_{i_l}(y_{j_l})T_{i_h}(y_{j_h})T_{i_\eta}(y_{j_\eta}) = 0 \qquad \text{(B.6)}$$

for $i_z = 1, \ldots, n_z, z \in \{k, l, h, \eta\}, i_z \in \Upsilon$. Define the product of the Chebyshev polynomials for $z \in \{k, l, h, \eta\}$ evaluated at the Chebyshev nodes $(y_{j_k}, y_{j_l}, y_{j_h}, y_{j_\eta})$ as

$$\mathbf{T}_i(y_j) \equiv T_{i,k}(y_{j_k})T_{i,l}(y_{j_l})T_{i,h}(y_{j_h})T_{i,\eta}(y_{j_\eta}),$$

and define $\mathbf{J}$ as the updated solution of the Bellman equation in (B.5) as

$$\mathcal{J}(\mathbf{c}^\tau) = \begin{bmatrix} \hat{J}^*(y_{1_k}, y_{1_l}, y_{1_h}, y_{1_\eta}; \mathbf{c}^\tau) \\ \vdots \\ \hat{J}^*(y_{m_k}, y_{m_l}, y_{m_h}, y_{m_\eta}; \mathbf{c}^\tau). \end{bmatrix}.$$

We will write these conditions in matrix form as

$$\mathcal{T} = \begin{bmatrix} \mathbf{T}_0(y_1) \ldots \mathbf{T}_0(y_{m_z}) \\ \vdots \ldots \vdots \\ \mathbf{T}_{n_z}(y_1) \ldots \mathbf{T}_{n_z}(y_{m_z}) \end{bmatrix}.$$

Using these definitions, we can write the orthogonality conditions in (B.5) in matrix form as

$$\mathcal{T}\mathcal{T}'\mathbf{c}^{\tau+1} = \mathcal{T}\mathcal{J}(\mathbf{c}^{\tau}),$$

which implies a new estimate of the Chebyshev coefficients $\mathbf{c}^{\tau+1}$ as a function of the coefficients $\mathbf{c}^{\tau}$ as

$$\mathbf{c}^{\tau+1} = (\mathcal{T}\mathcal{T}')^{-1}\mathcal{T}\mathcal{J}(\mathbf{c}^{\tau}). \tag{B.7}$$

## B.3 Numerical accuracy of the algorithm

Consider the Bellman equation that the transformed value function satisfies for the power-power specification of ambiguity averse preferences.

$$\hat{J}(\hat{k}_t, \mu_t) = \left\{ \frac{(\hat{c}_t^{\nu} l_t^{1-\nu})^{1-\gamma}}{1-\gamma} + \beta \left[ E_{\mu_t} \left( E_{x_t} \left( \hat{J}(\hat{k}_{t+1}, \mu_{t+1}) \exp(\nu(1-\gamma) g_{A,t+1}) \right) \right)^{1-\alpha} \right]^{\frac{1}{1-\alpha}} \right\}.$$

Since the value function is assumed to be homogeneous of order $\nu(1-\gamma)$, we can define

$$j(\hat{k}_t, \mu_t) = \left[ \nu(1-\gamma)\hat{J}(\hat{k}_t, \mu_t) \right]^{\frac{1}{\nu(1-\gamma)}}.$$

By definition, $j(\hat{k}_t, \mu_t)$ should be on the order of wealth (capital). The value function satisfies

$$\begin{aligned} \hat{j}(\hat{k}_t, \mu_t) &= \left\{ \nu(1-\gamma) \left( \frac{(\hat{c}_t^{\nu} l_t^{1-\nu})^{1-\gamma}}{1-\gamma} + \beta \left[ E_{\mu_t} \left( E_{x_t} \left( \hat{J}(\hat{k}_{t+1}, \mu_{t+1}) \exp(\nu(1-\gamma) g_{A,t+1}) \right) \right)^{1-\alpha} \right]^{\frac{1}{1-\alpha}} \right)^{\frac{1}{\nu(1-\gamma)}} \right\} \\ &\equiv RHS. \end{aligned}$$

Now compute the error

$$\epsilon = \left[ \hat{j}(\hat{k}_t, \mu_t) - RHS \right] / c_t,$$

where $\epsilon$ is the intertemporal mistake that we make by using the approximation to the value function expressed in consumption units. This error is evaluated on a grid outside of the original grid used to calculate the approximation to the value function. Following Judd (1998), we can report measures such as $E_1 = \log_{10}(\max(|\epsilon|))$ and $E_2 = \log_{10}(\text{mean}(|\epsilon|))$. A value of -4 for the first indicator means that by using the rule, a consumer would make $1 error out of every $10,000 spent.

Table B.1: Accuracy of the Numerical Solution

| | | | I. Baseline Model | | II. Panel A | | II. Panel B | |
|---|---|---|---|---|---|---|---|---|
| $\gamma$ | $\alpha$ | $\rho$ | $E_1$ | $E_2$ | $E_1$ | $E_2$ | $E_1$ | $E_2$ |
| 2 | 5 | 0.30 | -3.8837 | -3.9494 | -3.9870 | -4.0501 | -3.9039 | -4.0584 |
| 2 | 5 | 0.85 | -3.8817 | -3.9453 | -3.9904 | -4.0536 | -3.9825 | -4.0595 |

*Note*: As in the earlier simulations corresponding to the business cycle moments, the decision rules under each parameter configuration are computed on the assumption that agents do not observe the persistence parameter, $\rho_k$, governing the temporary component of the TFP process while accuracy measures are computed for simulations based on draws from the TFP process for a given persistence parameter, $\rho_k$.

Table B.1 shows the values of $\epsilon$ for the different parameterizations considered in this study. The approximation error for the specifications with $\alpha > \gamma > 1$ all perform reasonably well, and they are similar to the values reported by Collard *et al.* (2016) for their unknown persistence case.

# C Bayesian estimation of the TFP processes

Given the underlying model described in the text, we assume that the agent cannot infer on the true data generating process (DGP) but she assumes that it can be either a model with low persistence or a high persistence. Since identification is the major challenge and plays a central role in our model and in agents' behavior, we use simulation-based Bayesian inference with noninformative priors to estimate the model parameters. Specifically, we use Gibbs sampling together with data augmentation (see Geman and Geman, 1984; Tanner and Wong, 1987) to obtain posterior results. Since the model is a special case of the unobserved components model with Gaussian distributions, we use the Kalman filter together with a simulation smoother. For the simulation smoother, we use the smoother proposed in Carter and Kohn (1994) and Frühwirth-Schnatter (1994). The resulting simulation scheme at the $m^{th}$ step is as follows.

1. Sample $\rho$ from $p(\rho|x_{0:T}^{(m-1)}, \sigma_y^2, \sigma_x^2, \bar{g})$.
2. Sample $\sigma_y^2$ from $p(\rho|x_{0:T}^{(m-1)}, \rho, \sigma_x^2, \bar{g})$.
3. Sample $\sigma_x^2$ from $p(\rho|x_{0:T}^{(m-1)}, \rho, \sigma_y^2, \bar{g})$.
4. Sample $\bar{g}$ from $p(\rho|x_{0:T}^{(m-1)}, \rho, \sigma_x^2, \sigma_y^2, \bar{g})$.
5. Sample $x_{0:T}^{(m-1)}$, from $p(x_{0:T}|\rho, \sigma_y^2, \sigma_x^2, \bar{g})$ using Kalman filter and a simulation smoother.

While using noninformative priors reduces the posterior results to be identical with a pure likelihood based inference, it also provides us the entire distribution of the model parameters. Examining the distribution of, most notably, the persistence parameter is an integral part of the approach followed in this paper in the sense that it provides a rational for agent's ignorance regarding the persistence of the TFP growth process. The unrestricted model estimates are displayed in Table C.2 and the distributions of the parameters are displayed in Figure C.1. To evaluate the model, we use both the maximum likelihood and marginal likelihood values. Marginal likelihood can be computed as

$$p(y_{1:T}|M) = \int_{\theta} p(y_{1:T}|\theta)p(\theta)d\theta \tag{C.1}$$

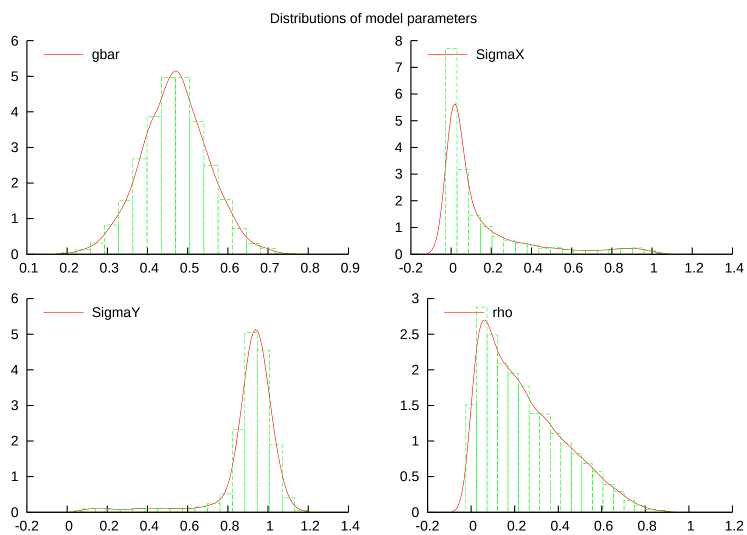
where $\theta = (\rho, \sigma_y^2, \sigma_x^2, \bar{g})$. As the marginal likelihood is computed by integrating out the (prior) parameter distributions, it provides a robust way of computing a performance measure of the model. To compute the integral we use the modified harmonic mean estimator of Geweke (1999).

Table C.2: Posterior results for the model using the TFP-util data for the period 1947-2:1977:4

| | | | |
|---|---|---|---|
| $\bar{g}$ | 0.468 (0.083) | $\sigma_x$ | 0.147 (0.231) |
| $\sigma_g$ | 0.902 (0.161) | $\rho$ | 0.243 (0.187) |
| Maximum likelihood | -167.40 | | |
| Marginal likelihood | -170.06 | | |

*Note*: The results show the posterior means and posterior standard deviations (in parenthesis) of the model parameters of the state space described by equation (3.1) in the text. The inference carried out 60,000 draws where the first 10,000 are used as the burn-in sample. We kept every $5^th$ draw, which yields a sample of 10,000 draws from the ergodic distribution.

Figure C.1: Distribution of parameters in the unrestricted model using TFP-util.




From Figure C.2 and C.3 which show the distributions for the estimated processes with $\rho = 0.30$ and $\rho = 0.85$ in Table 2 in the text, we see that imposing a high persistence smooths the short-tun TFP process as expected, while we observe more erratic changes when we impose low persistence. However, both process achieve a very similar maximum likelihoods and marginal likelihood values indicating the agent's ignorance about the true underlying DGP governing the TFP growth process.

Figure C.2: Distribution of parameters in the unrestricted model using TFP-util with $\rho = 0.30$ for the sample 1947-2:1977-4.

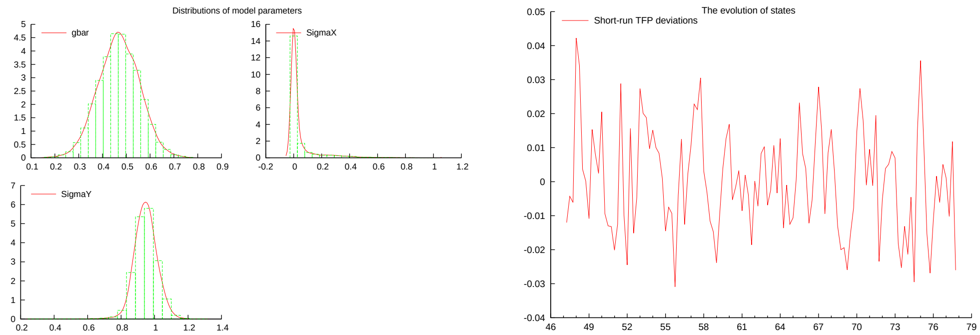
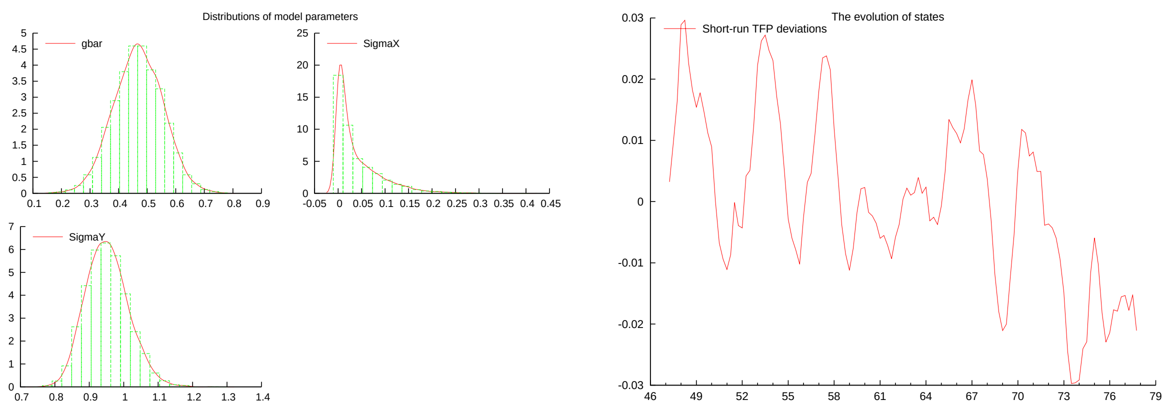



Figure C.3: Distribution of parameters in the unrestricted model using TFP-util with $\rho = 0.85$.




## C.1 Informative priors

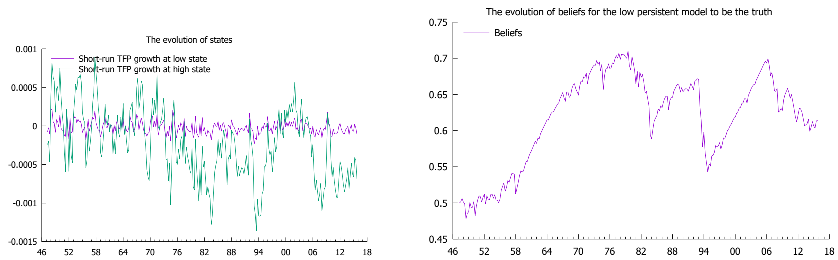
To have a more refined view of the impact of information and learning, we also generate estimates of the TFP process using more informative priors. We may view more informative prior distributions as a measure of the agent's "confidence" regarding the nature of the underlying TFP process. The priors for variances are set as almost degrees of freedom (dof) observations each with variance Scale/dof. The results of the estimation are described in Table C.3. The results in Panel A through Panel F of this table are generated by assuming progressively more informative priors compared to the uninformative prior case underlying the results in Table 4 in the text.

Table C.3: Posterior results for the model using TFP-util for different values of $\rho$ using the sample of 1947-2 : 1977-4.

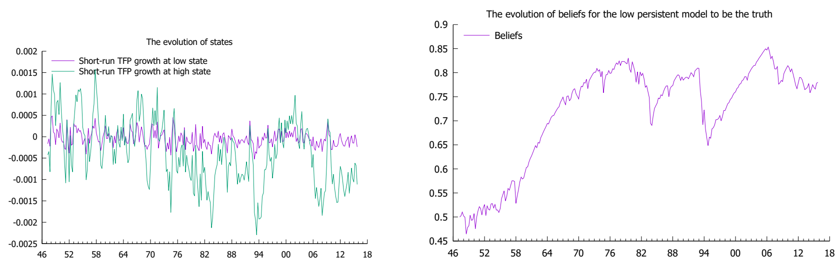
| | Prior | Posterior | |
|---|---|---|---|
| $\rho$ | | 0.300 | 0.850 |
| PANEL A | | | |
| $\bar{g}$ | – | 0.466 (0.083) | 0.466 (0.084) |
| $\sigma_g$ | IG(dof:2$\frac{1}{16}$, Scale: dof$*$0.01) | 0.925 (0.072) | 0.940 (0.062) |
| $\sigma_x$ | IG(dof:2$\frac{1}{16}$, Scale: dof$*$0.01) | 0.159 (0.109) | 0.100 (0.037) |
| Maximum Likelihood | | -167.40 | -167.52 |
| Marginal Likelihood | | -170.08 | -170.55 |
| PANEL B | | | |
| $\bar{g}$ | – | 0.466 (0.083) | 0.466 (0.081) |
| $\sigma_g$ | IG(dof:2$\frac{1}{16}$, Scale: dof$*$0.03) | 0.905 (0.081) | 0.936 (0.062) |
| $\sigma_x$ | IG(dof:2$\frac{1}{16}$, Scale: dof$*$0.03) | 0.237 (0.127) | 0.139 (0.037) |
| Maximum Likelihood | | -167.41 | -167.71 |
| Marginal Likelihood | | -170.23 | -171.39 |
| PANEL C | | | |
| $\bar{g}$ | – | 0.466 (0.078) | 0.466 (0.084) |
| $\sigma_g$ | IG(dof:2$\frac{1}{16}$, Scale: dof$*$0.05) | 0.891 (0.089) | 0.933 (0.063) |
| $\sigma_x$ | IG(dof:2$\frac{1}{16}$, Scale: dof$*$0.05) | 0.282 (0.135) | 0.162 (0.044) |
| Maximum Likelihood | | -167.42 | -167.86 |
| Marginal Likelihood | | -170.34 | -172.02 |
| PANEL D | | | |
| $\bar{g}$ | – | 0.466 (0.078) | 0.466 (0.083) |
| $\sigma_g$ | IG(dof:2$\frac{1}{16}$, Scale: dof$*$0.10) | 0.867 (0.098) | 0.928 (0.064) |
| $\sigma_x$ | IG(dof:2$\frac{1}{16}$, Scale: dof$*$0.10) | 0.349 (0.139) | 0.201 (0.049) |
| Maximum Likelihood | | -167.43 | -168.16 |
| Marginal Likelihood | | -170.50 | -173.10 |
| PANEL E | | | |
| $\bar{g}$ | – | 0.467 (0.074) | 0.466 (0.082) |
| $\sigma_g$ | IG(dof:2$\frac{1}{16}$, Scale: dof$*$0.30) | 0.819 (0.105) | 0.915 (0.067) |
| $\sigma_x$ | IG(dof:2$\frac{1}{16}$, Scale: dof$*$0.30) | 0.467 (0.134) | 0.283 (0.057) |
| Maximum Likelihood | | -167.51 | -169.08 |
| Marginal Likelihood | | -170.67 | -175.54 |
| PANEL F | | | |
| $\bar{g}$ | – | 0.467 (0.072) | 0.466 (0.082) |
| $\sigma_g$ | IG(dof:2$\frac{1}{16}$, Scale: dof$*$0.50) | 0.796 (0.105) | 0.907 (0.068) |
| $\sigma_x$ | IG(dof:2$\frac{1}{16}$, Scale: dof$*$0.50) | 0.522 (0.126) | 0.333 (0.061) |
| Maximum Likelihood | | -167.56 | -169.79 |
| Marginal Likelihood | | -170.78 | -177.06 |

*Note*: The results show the posterior means and posterior standard deviations (in parenthesis) of the model parameters in the state space model displayed by equation (3.1) in the text. The inference is carried out 60,000 draws where the first 10,000 are used as burn-in sample. We kept every $5^th$ draw, which yields a sample of 10,000 draws from the ergodic distribution.

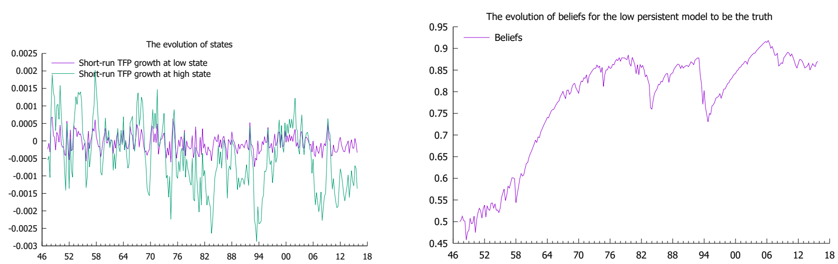
Figure C.4: Evolution of the agent's beliefs for the low persistence model to be the true DGP over the sample 1978-1:2015-4.



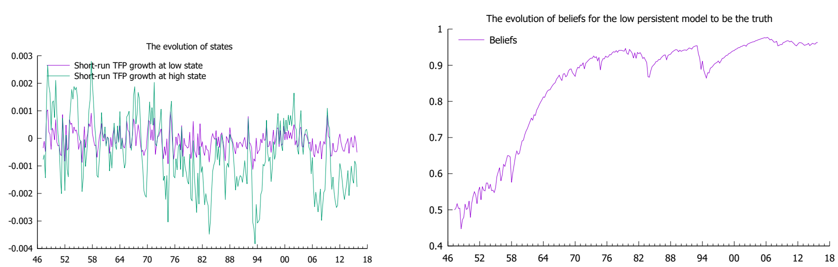

PANEL A



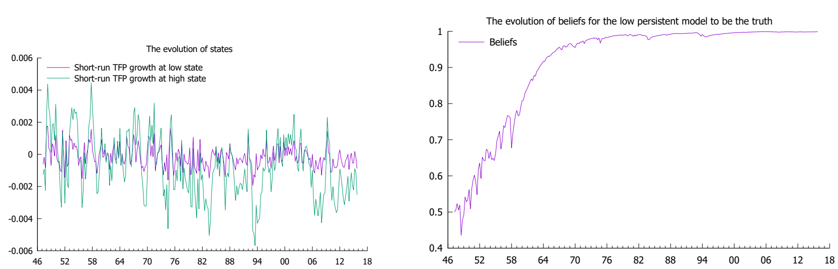

PANEL B



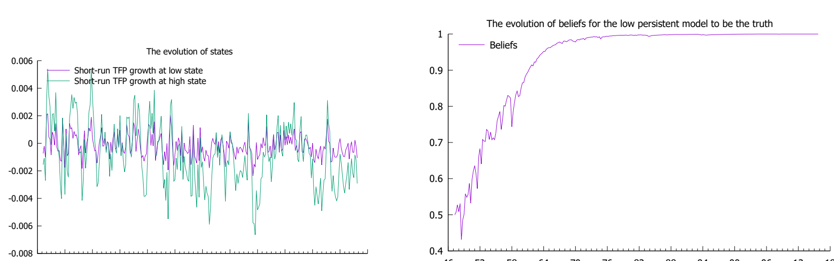

PANEL C




PANEL D




PANEL E




PANEL F

## C.2 Industry estimates

In this section we present posterior results for 9 representative industries based on the data set from Bloom *et al.* (2012). The data are from the replication file (stata data file) of the Bloom *et al.* (2012) paper. The data span 1971-2009 with annual frequency, and the estimates are based on TFP data expressed in percent. Hence, the parameters (except the $\rho$'s) are divided by $100 \times 4$ in the computation of the agent's decision rule at the quarterly frequency. Below are the results for selected 4-digit SIC industries from group 24, 28, 29, 30, 33, 34, 35, 36, and 37. Our calculations are limited by the existence of sectoral data for which there exist significant missing observations throughout the entire sample period. Specifically, we dropped industries with more than 5 missing observations. We estimated TFP processes for 3 industries from each of these sectors but due to space limitations, we only report results for one industry from each group.

Table C.4: Posterior results for the model using data on sectoral TFP for selected industries using the sample of 1947-2:1977-4

| | 2421 Sawmills and Planing Mills, General | | 2869 Industrial Organic Chemicals | | 2911 Petroleum Mining | |
|---|---|---|---|---|---|---|
| | 0.30 | 0.85 | 0.30 | 0.85 | 0.30 | 0.85 |
| $\bar{g}$ | 0.718 (0.083) | 0.719 (0.098) | 0.5171 (0.0862) | 0.5171 (0.0963) | -0.449 (0.095) | -0.449 (0.099) |
| $\sigma_g$ | 1.767 (1.233) | 2.978 (0.503) | 1.589 (0.864) | 2.273 (0.328) | 4.830 (1.826) | 5.667 (0.717) |
| $\sigma_x$ | 2.252 (1.179) | 0.976 (0.623) | 1.346 (0.888) | 0.513 (0.379) | 2.033 (2.066) | 0.808 (0.694) |
| Max. | -98.35 | -98.74 | -83.93 | -84.08 | -123.24 | -123.24 |
| Mar. | -100.12 | -101.64 | -85.46 | -86.36 | -125.16 | -125.22 |

| | 3084 Plastic Pipes | | 3315 Steel Wiredrawing and Steel Nails and Spikes | | 3448 Prefabricated Metal Buildings and Components | |
|---|---|---|---|---|---|---|
| | 0.30 | 0.85 | 0.30 | 0.85 | 0.30 | 0.85 |
| $\bar{g}$ | -0.100 (0.100) | -0.100 (0.100) | -0.495 (0.079) | -0.494 (0.096) | 0.5469 (0.0872) | 0.5451 (0.0987) |
| $\sigma_g$ | 6.140 (0.899) | 6.292 (0.761) | 2.028 (1.916) | 4.141 (1.558) | 2.9253 (2.0054) | 4.8793 (0.7413) |
| $\sigma_x$ | 1.130 (1.254) | 0.692 (0.587) | 3.805 (1.753) | 1.851 (1.749) | 3.0347 (1.9725) | 0.8328 (0.8480) |
| Max. | -123.35 | -123.36 | -116.87 | -118.45 | -111.42 | -111.67 |
| Mar. | -125.08 | -126.00 | -118.71 | -120.77 | -112.80 | -114.17 |

| | 3519 Internal combustion engines | | 3621 Motors and generators | | 3731 Ship building and repairing | |
|---|---|---|---|---|---|---|
| | 0.30 | 0.85 | 0.30 | 0.85 | 0.30 | 0.85 |
| $\bar{g}$ | -0.536 (0.093) | -0.536 (0.100) | -0.259 (0.090) | -0.259 (0.099) | 0.1422 (0.0987) | 0.1421 (0.0989) |
| $\sigma_g$ | 3.919 (1.841) | 5.148 (0.662) | 2.680 (1.489) | 3.907 (0.580) | 5.1789 (0.8666) | 5.3449 (0.6645) |
| $\sigma_x$ | 2.351 (1.991) | 0.730 (0.705) | 2.201 (1.555) | 0.714 (0.702) | 0.9997 (1.1156) | 0.6303 (0.5186) |
| Max. | -119.14 | -119.15 | -108.93 | -109.09 | -113.97 | -113.97 |
| Mar. | -120.91 | -121.93 | -110.56 | -112.48 | -115.80 | -116.16 |

*Note*: The results show the posterior means and posterior standard deviations (in parenthesis) of the model parameters the state space described by equation (3.1) in the text (evaluated in percentage terms). The inference was carried out with 60,000 draws where the first 10,000 are used as burn-in sample. We kept every $5^t h$ draw, which yields a sample of 10,000 draws from the ergodic distribution.

We also calculated estimates for 6 industries with some of the largest uncertainty measures on TFP calculated according to the measure in Bloom *et al.* (2012). These are (in descending order) 2075, 2911, 3571, 2022, 2048, 2879, 2874. We could not get the decision rules to converge under the estimates for Industry 2874, which has very high variances for the shocks $\epsilon_{A_k,t}$ and $\epsilon_{x_k,t}$ for $k = l, h$. We already have the results for Industry 2911 in Table C.4. In Table C.5, we display the results for 3 of the remaining industries, 3571, 2075 and 2879. Finally, we also estimate the model with industry 3221 in Table C.6 as the industry with the lowest degree of uncertainty on TFP.

Table C.5: Posterior results for the model using sectoral TFP data for high uncertainty indices using the sample of 1947-2:1977-4

| | 3571 Soybean oil mills | | 2075 Electronic computers | | 2879 Pesticides and agricultural chemicals, not elsewhere classified | |
|---|---|---|---|---|---|---|
| | 0.30 | 0.85 | 0.30 | 0.85 | 0.30 | 0.85 |
| $\bar{g}$ | 17.255 (0.095) | 17.256 (0.100) | 1.702 (0.100) | 1.702 (0.100) | -0.234 (0.086) | -0.234 (0.100) |
| $\sigma_g$ | 7.420 (3.560) | 9.412 (1.341) | 11.408 (1.461) | 11.637 (1.399) | 4.641 (4.288) | 10.025 (1.810) |
| $\sigma_x$ | 4.528 (3.908) | 1.946 (1.562 | 1.516 (1.808) | 0.946 (0.862) | 7.341 (4.021) | 2.035 (2.378) |
| Max. | -140.68 | -140.39 | -146.16 | -146.16 | -141.81 | -142.99 |
| Mar. | -142.44 | -142.76 | -148.17 | -148.17 | -143.49 | -145.57 |

*Note*: The results show the posterior means and posterior standard deviations (in parenthesis) of the model parameters the state space described by equation (3.1) in the text (evaluated in percentage terms). The inference was carried out with 60,000 draws where the first 10,000 are used as burn-in sample. We kept every $5^th$ draw, which yields a sample of 10,000 draws from the ergodic distribution.

Table C.6: Posterior results for the model using sectoral TFP data for high uncertainty indices using the sample of 1947-2:1977-4

| | 3221 Glass containers | | | | | |
|---|---|---|---|---|---|---|
| | 0.30 | 0.85 | 0.30 | 0.85 | 0.30 | 0.85 |
| $\bar{g}$ | 0.042 (0.095) | 0.042 (0.098) | | | | |
| $\sigma_g$ | 2.489 (0.697) | 2.763 (0.356 | | | | |
| $\sigma_x$ | 1.011 (0.872) | 0.489 (0.356) | | | | |
| Max. | -93.10 | -93.09 | | | | |
| Mar. | -95.09 | -95.56 | | | | |

*Note*: The results show the posterior means and posterior standard deviations (in parenthesis) of the model parameters of the state space described by equation (3.1) in the text (evaluated in percentage terms). The inference was carried out with 60,000 draws where the first 10,000 are used as burn-in sample. We kept every $5^th$ draw, which yields a sample of 10,000 draws from the ergodic distribution.

# D A log-linear approximation

In Tallarini (2000) and Backus *et al.* (2014), the impact of uncertainty aversion and ambiguity aversion is examined in the context of recursive preferences that admit a separation between intertemporal substitution and risk aversion or equivalently, among intertemporal substitution, risk aversion, and ambiguity aversion. These papers derive log-linear decision rules for capital and consumption, and show that the parameter of risk aversion (ambiguity aversion) does not directly affect the coefficient on the current capital stock in the log-linear decision rules for the capital stock and consumption. Thus, they show that the parameter of risk aversion (ambiguity aversion) plays no role in the endogenous dynamics of the capital stock.

Here we show this result in the one-process model considered in the text. This may be viewed as the special case in which beliefs converge to a given process such that $\eta_t = 1$ (or equivalently, $\eta_t = 0$). We also omit the labor-leisure choice because it does not add to the substantive analysis. The social planner's problem in this environment is expressed as

$$\hat{J}(\hat{k}_t, \hat{x}_t) = \max_{\hat{c}_t, \hat{i}_t} \left\{ \frac{\hat{c}_t^{\,1-\gamma}}{1-\gamma} + \beta \left[ E_{\hat{x}_t} \left( E_{x_t} \hat{J}(\hat{k}_{t+1}, \hat{x}_{t+1}) \exp((1-\gamma) g_{A,t+1}) \right)^{1-\alpha} \right]^{\frac{1}{1-\alpha}} \right\}. \quad \text{(D.1)}$$

subject to the resource constraint and the laws of motion for beliefs as

$$\hat{c}_t + \exp(g_{A,t+1}) \hat{k}_{t+1} \leq \hat{k}_t^a + (1-\delta)\hat{k}_t \equiv f(k_t),$$

$$\hat{x}_{t+1} = \rho \hat{x}_t + K \nu_{t+1},$$

$$\nu_{t+1} = g_{A,t+1} - \bar{g} - \rho \hat{x}_t = \rho(x_t - \hat{x}_t) + \sigma_x \epsilon_{x,t+1} + \sigma_A \epsilon_{A,t+1}.$$

where $K = \rho \Omega f^{-1}$, $f = \Omega + \sigma_A^2$, $\Omega = \rho^2 \Omega - \rho^2 \Omega^2 f^{-1} + \sigma_x^2$, such that $f = E[(g_{A,t+1} - E(g_{A,t+1}))^2 | g_{A,1}, \ldots, g_{A,t}]$ and $\Omega = E[(x_{t+1} - \hat{x}_{t+1})^2 | g_{A,1}, \ldots, g_{A,t}]$, $k = l, h$.

The state variables of the model are defined as $(\hat{k}_t, \hat{x}_t)$. As before, the approach to generating log-linear solutions is to approximate the equilibrium relations around the mean log of the steady values. Substituting for the resource constraint into the logarithm of the indirect value function to obtain

$$\log \hat{J}(\hat{k}_t, \hat{x}_t) = \max_{\hat{c}_t} \log \left\{ \frac{\hat{c}_t{}^{1-\gamma}}{1-\gamma} + \beta \left[ E_{\hat{x}_t} \left( E_{x_t} \hat{J}(\hat{k}_{t+1}, \hat{x}_{t+1}) \exp((1-\gamma) g_{A,t+1}) \right)^{1-\alpha} \right]^{\frac{1}{1-\alpha}} \right\}.$$

Define

$$\Phi_t = \left[ E_{\hat{x}_t} \left( E_{x_t} \hat{J}(\hat{k}_{t+1}, \hat{x}_{t+1}) \exp((1-\gamma) g_{A_{t+1}}) \right)^{1-\alpha} \right],$$

and

$$\Psi_t = \left\{ E_{\hat{x}_t} \left[ \left( E_{x_t} \hat{J}(\hat{k}_{t+1}, \hat{x}_{t+1}) \exp((1-\gamma) g_{A_{t+1}}) \right)^{-\alpha} E_{x_t} \hat{J}_k(\hat{k}_1, \hat{x}_{t+1}) \exp(-\gamma g_{A_{t+1}}) \right] \right\}$$

The first-order condition with respect to $\hat{c}_t$ is

$$0 = \hat{c}_t^{-\gamma} - \beta \Phi_t^{\frac{\alpha}{1-\alpha}} \Psi_t, \tag{D.2}$$

where we have omitted the multiplication of the entire right-side term by $1/J_t$. The envelope condition is given by

$$\hat{J}_{kt} = \beta \Phi_t^{\frac{\alpha}{1-\alpha}} \Psi_t f_{kt}, \tag{D.3}$$

where again we have cancelled the terms $1/J_t$ on both sides of the equation and $f_{kt}$ is the marginal product of current capital..

From these two equations, we infer that

$$\hat{c}_t^{-\gamma} = J_{kt}/f_{kt}. \tag{D.4}$$

This equation can be used to derive the decision for the consumption allocation. Evidently, it only depends on the derivative of the value function $J_{kt}$, as in the additive models discussed by Backus *et al.* (2014).

## D.1 Solving for the log-linear decision rules

As in Backus *et al.* (2014), we assume that the value function itself can be approximated as

$$\log(\hat{J}_t) = d(p_k \log(\hat{k}_t) + p_x \hat{x}_t), \tag{D.5}$$

where $d = (J - p_0)/J$ and $q_x$ is negative. [24] From this expression, we can evaluate $J_{kt}$ as

$$\log \hat{J}_k(\hat{k}_t, \hat{x}_t) = [p_k - 1]\log(\hat{k}_t) + p_x \hat{x}_t] = q_k \log(\hat{k}_t) + q_x \hat{x}_t,$$

and we use the approximation

$$\begin{aligned}
\log(f_k(k_t)) &= \frac{f_{kk}}{f_k} e^{\log(k)}(\log(k_t) - \log(k)) \\
&= \frac{f_{kk}k}{f_k} \log(k_t) + \text{constants} \\
&= \lambda_r \log(k_t) + \text{constants.}
\end{aligned}$$

Putting these two results yields

$$\begin{aligned}
\log(\hat{c}_t) &= -\gamma^{-1}\left(q_k \log(\hat{k}_t) + q_x \hat{x}_t\right) + \gamma^{-1}\lambda_r \log(\hat{k}_t) \\
&= -\gamma^{-1}(q_k - \lambda_r)\log(\hat{k}_t) - \gamma^{-1} q_x \hat{x}_t \\
&= h_{c0} + h_{ck}\log(\hat{k}_t) + h_{c,x}\hat{x}_t
\end{aligned}$$

We can also derive the controlled law of motion for the capital stock as before using the log-linearized resource constraint and the log-linear decision rule for consumption as

$$\begin{aligned}
\log(\hat{k}_{t+1}) &= \lambda_k \log(\hat{k}_t) - \lambda_c \log(\hat{c}_t) - g_{A,t+1} \\
&= \lambda_k \log(\hat{k}_t) + \lambda_c \gamma(q_k - \lambda_r)\log(\hat{k}_t) + \lambda_c \gamma^{-1}\left(q_x \hat{x}_t\right) - x_{t+1} - \sigma_A \epsilon_{A,t+1} \\
&= [\lambda_k + \lambda_c \gamma^{-1}(q_k - \lambda_r)]\log(\hat{k}_t) + \lambda_c \gamma^{-1} q_x \hat{x}_t - x_{t+1} - \sigma_A \epsilon_{A,t+1} \\
&= h_{k0} + h_{kk}\log(\hat{k}_t) + h_{k,x}\hat{x}_t - g_{A,t+1}.
\end{aligned}$$

[24] These authors derive this representation by initially assuming that the derivative of the value function is approximated log-linearly as

$$\log(J_{kt}) = p_1 + \log(p_k)(p_k - 1)\log(k_t) + p_x^T x_t + p_v v_t \quad \Rightarrow \quad J_{kt} = k_t^{p_k - 1}\exp(p_1 + p_X^T x_t + p_v v_t).$$

Integrating with respect to $k_t$ yields

$$J_t = p_0 + k_t^{p_k}\exp(p_1 + p_X^T x_t + p_v v_t).$$

This is not log-linear unless $p_0 = 0$. However, a log-linear approximation yields the representation in the text.

This representation is very similar to the one derived from the recursive non-expected utility model in Backus *et al.* (2014). For deterministic consumption paths, the smooth ambiguity model implies that $\gamma^{-1}$ is the intertemporal elasticity of substitution in consumption, which affects the decision rule for the optimal capital stock directly..

To complete the solution, we seek equations to determine $q_k$ and $q_x$. To find them, we need to evaluate the right-side of the envelope condition as before. Notice that

$$\log(\hat{J}_{kt}) = \log(\beta) + \frac{\alpha}{1-\alpha}\log(\Phi_t) + \log(f_{kt}) + \log(\Psi_t).$$

Using the results we have derived so far, we can evaluate the the terms on the right-side of the above equation as

$$\frac{\alpha}{1-\alpha}\log(\Phi_t) + \log(\Psi_t) = \frac{\alpha}{1-\alpha}\log\left(E_{\hat{x}_t}\left[E_{x_t}\left(\hat{J}_{t+1}\exp((1-\gamma)g_{A_{t+1}})\right)^{1-\alpha}\right]\right)$$
$$+\log\left(E_{\hat{x}_t}\left[E_{x_t}\left(\hat{J}_{t+1}\exp((1-\gamma)g_{A_{t+1}})\right)^{-\alpha} E_{x_t}\hat{J}_{k,t+1}\exp(-\gamma g_{A_{t+1}})\right]\right).$$

To simplify notation, let $E_t = E_{\hat{x}_t}E_{x_t}$. To calculate this explicitly, we make use of the following result as

$$\frac{\alpha}{1-\alpha}\log E_{\hat{x}_t}\left[E_{x_t}\left(\hat{J}_{t+1}\exp((1-\gamma)g_{A_{t+1}})\right)^{1-\alpha}\right] =$$
$$\alpha E_t\left[\left(\log(\hat{J}_{t+1}) + (1-\gamma)g_{A,t+1}\right)\right] + \text{variance terms}^{25}.$$

Likewise,

$$\log E_{\hat{x}_t}\left\{\left[E_{x_t}\left(\hat{J}_{t+1}\exp((1-\gamma)g_{A_{t+1}})\right)^{-\alpha}\right]E_{x_t}\left[\hat{J}_{k,t+1}\exp(-\gamma g_{A,t+1})\right]\right\} =$$
$$-\alpha E_t\left[\log(\hat{J}_{t+1}) + (1-\gamma)g_{A,t+1}\right] + E_t\left[\log(\hat{J}_{k,t+1}) - \gamma g_{A,t+1}\right] + \text{variance terms.}^{26}$$

[25]The variance terms are $(\alpha(1-\alpha)/2)Var_t\log(\hat{J}_{t+1}) + (\alpha(1-\alpha)(1-\gamma)^2/2)Var_t(g_{A,t+1})$.

Hence,

$$\frac{\alpha}{1-\alpha}\log(\Phi_t) + \log(\Psi_t) = \alpha E_t\left[\log(\hat{J}_{t+1} + (1-\gamma)g_{A_{t+1}}))\right]$$
$$-\alpha E_t\left[\log(\hat{J}_{t+1} + (1-\gamma)g_{A_{t+1}})\right] + E_t\left[\log(\hat{J}_{k,t+1}) - \gamma g_{A_{t+1}}\right] + \text{variance terms},$$

so that the impact of the ambiguity aversion parameter $\alpha$ cancels out completely, and does not affect the dynamics of the capital stock.

Thus, the coefficients of the capital stock rule are determined from the simplified envelope condition as

$$\log(\hat{J}_{k,t}) = E_t\left[(\log(\hat{J}_{k,t+1}) - \gamma g_{A_{t+1}})\right] + \log(f_k) + \text{variance terms},$$

which can be re-written as

$$\begin{aligned}
q_k \log(\hat{k}_t) + q_x\hat{x}_t &= \log E_t\left[(q_k\log(\hat{k}_{t+1}) + q_x\hat{x}_{t+1}) - \gamma g_{A,t+1}\right] + \lambda_r\log(\hat{k}_t) \\
&= \log E_t\left[q_k(h_{kk}\log(\hat{k}_t) + h_{k,x}\hat{x}_t) + q_x\hat{x}_{t+1} - \gamma g_{A,t+1}\right] + \lambda_r\log(\hat{k}_t) \\
&= \log E_t\Big\{q_k(h_{kk}\log(\hat{k}_t) + h_{k,x}\hat{x}_t) + q_x\rho\hat{x}_t + q_x K[\rho(x_t - \hat{x}_t) + \sigma_x\epsilon_{x,t+1} + \sigma_A\epsilon_{A,t+1}] \\
&\quad -\gamma(\bar{g} + \rho x_t + \sigma_x\epsilon_{x,t+1} + \sigma_A\epsilon_{A,t+1})\} + \lambda_r\log(\hat{k}_t) \\
&= (q_k h_{kk} + \lambda_r)\log(\hat{k}_t) + (q_k h_{k,x} + q_x\rho - \gamma\rho)\hat{x}_t + q_x^2K^2\rho^2\frac{\Omega}{2} + \left[q_x^2K^2 + \gamma^2\right]\frac{\sigma_x^2 + \sigma_A^2}{2}
\end{aligned}$$

Thus,

$$q_k = q_k h_{kk} + \lambda_r = q_k[\lambda_k + \lambda_c\gamma^{-1}(q_k - \lambda_r)], \tag{D.6}$$

which is independent of the ambiguity aversion parameter $\alpha$. Given $q_k$, we can solve for $q_x$ from the envelope condition provided above as

$$q_x = q_k h_{k,x} + q_x\rho - \gamma\rho = q_k\lambda_c\gamma^{-1}q_x + q_x\rho - \gamma\rho. \tag{D.7}$$

[26] The variance terms in this case are $(\alpha^2/2)Var_t\log(\hat{J}_{t+1}) + (\alpha^2(1-\gamma)^2/2)Var_t(g_{A,t+1}) + Var_t\log(\hat{J}_{k,t+1}) + (\gamma^2/2)Var_t(g_{A,t+1})$.

Notice that if $\rho = 0$ so that the unobserved component $x_{t+1}$ has no persistence, then $q_x = 0$.

We can write the coefficients on the capital stock in the log-linear representation for the future capital stock and consumption as

$$
\begin{aligned}
h_{k,k} &= [\lambda_k + \lambda_c \gamma^{-1}(q_k - \lambda_r)] \\
h_{c,k} &= -\gamma^{-1}(q_k - \lambda_r).
\end{aligned}
$$

As in Backus *et al.* (2014), the ambiguity aversion parameter does not affect the endogenous dynamics of the capital stock. Instead, it affects the decision rule for the optimal capital stock only through the constant variance terms. A similar result is derived by Tallarini (2000). Furthermore, a *separation* property also holds in that the coefficient of the capital stock – $h_{kk}$ – is also independent of agents' beliefs about the underlying unobserved state, $\hat{x}_t$. Likewise, the dynamics induced by agents' beliefs about the transitory component of productivity growth – $h_{k,x}$ and $h_{c,x}$ – are also independent of the ambiguity aversion parameter, $\alpha$.

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Table 4: Simulation Results: Ambiguity about the Persistence/Variability of the Unobserved Component of TFP Growth

| | $\gamma = 0.5, \alpha = 0.8$ | | | | | $\gamma = 0.5, \alpha = 5$ | | | | | $\gamma = 2, \alpha = 5$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Simulations conditional on $\rho_k = 0.85$ | | | | | | | | | | | | | | |
| | Standard deviations | | | | | | | | | | | | | | |
| | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | $p$ |
| | 1.322 | 0.594 | 6.013 | 0.656 | 0.775 | 1.328 | 0.598 | 6.038 | 0.659 | 0.779 | 1.052 | 0.689 | 3.092 | 0.262 | 0.806 |
| | Correlations | | | | | | | | | | | | | | |
| | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | $p$ |
| $y$ | 1.000 | 0.728 | 0.942 | 0.909 | 0.936 | 1.000 | 0.728 | 0.942 | 0.909 | 0.936 | 1.000 | 0.985 | 0.979 | 0.954 | 0.995 |
| $c$ | | 1.000 | 0.460 | 0.377 | 0.923 | | 1.000 | 0.459 | 0.376 | 0.923 | | 1.000 | 0.930 | 0.890 | 0.997 |
| $i$ | | | 1.000 | 0.994 | 0.765 | | | 1.000 | 0.994 | 0.765 | | | 1.000 | 0.995 | 0.954 |
| $h$ | | | | 1.000 | 0.705 | | | | 1.000 | 0.704 | | | | 1.000 | 0.921 |
| $p$ | | | | | 1.000 | | | | | 1.000 | | | | | 1.000 |
| $i/y$ | 0.1694 | | | | | 0.1701 | | | | | 0.1576 | | | | |
| | Simulations conditional on $\rho_k = 0.30$ | | | | | | | | | | | | | | |
| | Standard deviations | | | | | | | | | | | | | | |
| | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | $p$ |
| | 1.307 | 0.583 | 5.957 | 0.650 | 0.763 | 1.304 | 0.583 | 5.949 | 0.649 | 0.762 | 1.034 | 0.676 | 3.042 | 0.258 | 0.791 |
| | Correlations | | | | | | | | | | | | | | |
| | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | $p$ | | | | | |
| $y$ | 1.000 | 0.728 | 0.943 | 0.911 | 0.936 | 1.000 | 0.727 | 0.943 | 0.911 | 0.936 | 1.000 | 0.986 | 0.979 | 0.955 | 0.995 |
| $c$ | | 1.000 | 0.463 | 0.381 | 0.922 | | 1.000 | 0.461 | 0.379 | 0.922 | | 1.000 | 0.931 | 0.891 | 0.997 |
| $i$ | | | 1.000 | 0.994 | 0.768 | | | 1.000 | 0.994 | 0.767 | | | 1.000 | 0.995 | 0.955 |
| $h$ | | | | 1.000 | 0.709 | | | | 1.000 | 0.707 | | | | 1.000 | 0.922 |
| $p$ | | | | | 1.000 | | | | | 1.000 | | | | | 1.000 |
| $i/y$ | 0.1722 | | | | | 0.1756 | | | | | 0.1600 | | | | |

*Note*: The model is simulated based on the decision rules for the main model with ambiguity where the agent cannot distinguish perfectly between two processes with persistence $\rho_l = 0.30$ and $\rho_h = 0.85$. The parameters characterizing the shock processes are derived from the estimates in Table 2 and assume that $\bar{g}_h = 0.00469$ and $\rho_h = 0.85$, $\sigma_{A_h} = 0.00952$, $\sigma_{x_h} = 0.0004$ for the high persistence process and $\rho_l = 0.30$, $\sigma_{A_l} = 0.00946$, $\sigma_{x_l} = 0.00044$ for the low persistence process.

Table 5: Simulation Results: Known Persistence/Variability of the Unobserved Component of TFP Growth

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Single process with $\rho = 0.85$ | | | | | | | | | | |
| | $\gamma = 0.5, \alpha = 0.8$ | | | | | $\gamma = 2, \alpha = 5$ | | | | |
| Standard deviations | | | | | | | | | | |
| | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | $p$ |
| | 1.162 | 0.639 | 3.272 | 0.437 | 0.754 | 0.946 | 0.822 | 1.532 | 0.113 | 0.835 |
| Correlations | | | | | | | | | | |
| | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | $p$ |
| $y$ | 1.000 | 0.968 | 0.981 | 0.958 | 0.986 | 1.000 | 0.999 | 0.998 | 0.985 | 0.999 |
| $c$ | | | 0.904 | 0.857 | 0.996 | | 1.000 | 0.996 | 0.980 | 1.000 |
| $i$ | | | 1.000 | 0.994 | 0.937 | | | 1.000 | 0.994 | 0.996 |
| $h$ | | | | 1.000 | 0.898 | | | | 1.000 | 0.981 |
| $p$ | | | | | 1.000 | | | | | 1.000 |
| $i/y$ | 0.2087 | | | | | 0.1774 | | | | |
| Single process with $\rho = 0.30$ | | | | | | | | | | |
| | $\gamma = 2, \alpha = 5$ | | | | | $\gamma = 0.5, \alpha = 0.8$ | | | | |
| Standard deviations | | | | | | | | | | |
| | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | $p$ |
| | 1.154 | 0.639 | 3.263 | 0.436 | 0.748 | 0.927 | 0.807 | 1.552 | 0.134 | 0.819 |
| Correlations | | | | | | | | | | |
| | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | $p$ |
| | 1.000 | 0.969 | 0.981 | 0.958 | 0.986 | 1.000 | 0.995 | 0.972 | 0.834 | 0.996 |
| | | 1.000 | 0.904 | 0.857 | 0.996 | | 1.000 | 0.945 | 0.777 | 1.000 |
| | | | 1.000 | 0.994 | 0.937 | | | 1.000 | 0.940 | 0.947 |
| | | | | 1.000 | 0.898 | | | | 1.000 | 0.781 |
| | | | | | 1.000 | | | | | 1.000 |
| $i/y$ | 0.2100 | | | | | 0.1767 | | | | |

*Note*: The model is simulated based on the decision rules for the model where the agent know with certainty the persistence of the unobserved component of TFP growth, namely, $x_{k,t}$ for $k = 1, 2$. The parameters characterizing the shock processes are derived from the estimates in Table 2 and assume that $\bar{g}_h = 0.00469$ and $\rho_h = 0.85$, $\sigma_{A_h} = 0.00952$, $\sigma_{x_h} = 0.0004$ for the high persistence process and $\rho_l = 0.30$, $\sigma_{A_l} = 0.00946$, $\sigma_{x_l} = 0.00044$ for the low persistence process.

Table 6: Simulation Results: Informative Priors about the Persistence/Variability of the Unobserved Component of TFP

| | Panel A | | | | | | | | | | Panel B | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Simulations conditional on $\rho_k = 0.85$ | | | | | | | | | | | | | | | | | | | |
| | $\gamma = 0.5, \alpha = 0.8$ | | | | | $\gamma = 2, \alpha = 5$ | | | | | $\gamma = 0.5, \alpha = 0.8$ | | | | | $\gamma = 2, \alpha = 5$ | | | | |
| | Standard deviations | | | | | | | | | | | | | | | | | | | |
| | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | |
| | 1.330 | 0.622 | 5.779 | 0.620 | 0.808 | 1.048 | 0.734 | 2.778 | 0.221 | 0.836 | 1.324 | 0.649 | 5.365 | 0.560 | 0.841 | 1.044 | 0.788 | 2.463 | 0.183 | 0. |
| | Correlations | | | | | | | | | | | | | | | | | | | |
| | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | |
| $y$ | 1.000 | 0.784 | 0.943 | 0.909 | 0.948 | 1.000 | 0.993 | 0.985 | 0.967 | 0.998 | 1.000 | 0.854 | 0.950 | 0.917 | 0.964 | 1.000 | 0.995 | 0.985 | 0.961 | 0. |
| $c$ | | 1.000 | 0.537 | 0.455 | 0.941 | | 1.000 | 0.959 | 0.930 | 0.999 | | 1.000 | 0.652 | 0.577 | 0.962 | | 1.000 | 0.964 | 0.930 | 0. |
| $i$ | | | 1.000 | 0.994 | 0.789 | | | 1.000 | 0.995 | 0.972 | | | 1.000 | 0.994 | 0.833 | | | 1.000 | 0.993 | 0. |
| $h$ | | | | 1.000 | 0.947 | | | | 1.000 | 0.729 | | | | 1.000 | 0.777 | | | | 1.000 | 0. |
| $p$ | | | | | 1.000 | | | | | 1.000 | | | | | 1.000 | | | | | 1. |
| $i/y$ | 0.1682 | | | | | 0.1582 | | | | | 0.1645 | | | | | 0.1580 | | | | |
| | Simulations conditional on $\rho_k = 0.30$ | | | | | | | | | | | | | | | | | | | |
| | $\gamma = 0.5, \alpha = 0.8$ | | | | | $\gamma = 2, \alpha = 5$ | | | | | $\gamma = 0.5, \alpha = 0.8$ | | | | | $\gamma = 2, \alpha = 5$ | | | | |
| | Standard deviations | | | | | | | | | | | | | | | | | | | |
| | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | |
| | 1.283 | 0.579 | 5.696 | 0.616 | 0.762 | 1.015 | 0.687 | 2.824 | 0.232 | 0.794 | 1.280 | 0.584 | 5.565 | 0.598 | 0.770 | 1.023 | 0.698 | 2.823 | 0.230 | 0. |
| | Correlations | | | | | | | | | | | | | | | | | | | |
| | $y$ | $c$ | $i$ | $h$ | $p$ | $y$ | $c$ | $i$ | $h$ | $p$ | | | | | | | | | | |
| $y$ | 1.000 | 0.765 | 0.946 | 0.914 | 0.945 | 1.000 | 0.991 | 0.984 | 0.964 | 0.997 | 1.000 | 0.789 | 0.948 | 0.916 | 0.950 | 1.000 | 0.990 | 0.983 | 0.962 | 0. |
| $c$ | | 1.000 | 0.518 | 0.438 | 0.934 | | 1.000 | 0.948 | 0.915 | 0.998 | | 1.000 | 0.557 | 0.479 | 0.941 | | 1.000 | 0.947 | 0.913 | 0. |
| $i$ | | | 1.000 | 0.995 | 0.789 | | | 1.000 | 0.995 | 0.966 | | | 1.000 | 0.995 | 0.803 | | | 1.000 | 0.995 | 0. |
| $h$ | | | | 1.000 | 0.731 | | | | 1.000 | 0.949 | | | | 1.000 | 0.747 | | | | 1.000 | 0. |
| $p$ | | | | | 1.000 | | | | | 1.000 | | | | | 1.000 | | | | | 1. |
| $i/y$ | 0.1694 | | | | | 0.1589 | | | | | 0.1719 | | | | | 0.1618 | | | | |

*Note*: The model is simulated based on the decision rules for the main model with ambiguity where the agent cannot distinguish perfectly between two processes persistence $\rho_l = 0.30$ and $\rho_h = 0.85$. In Panel A, the parameters of the technology shock process are set so that $\bar{g}_h = 0.00469$ and $\rho_h = 0.85$, $\sigma_{A_h} = 0.0094$, $\sigma_{x_h} =$ 0 for the high persistence process and $\rho_l = 0.30$, $\sigma_{A_l} = 0.00925$, $\sigma_{x_l} = 0.00159$ for the low persistence process. In Panel B, the parameters of the technology shock pr are set so that $\rho_h = 0.85$, $\sigma_{A_h} = 0.00936$, $\sigma_{x_h} = 0.00139$ for the high persistence process and $\rho_l = 0.30$, $\sigma_{A_l} = 0.00905$, $\sigma_{x_l} = 0.00237$ for the low persistence proce

Table 7: Simulation Results for the Risk-free Rate, Quarterly Rate

| Model | Industry name | Risk-free rate $r^f$ | $\tilde{X}$ | $SD(\tilde{X})$ | Uncertainty measure |
|---|---|---|---|---|---|
| Baseline model | - | 1.89 | 5.77e-05 | 0.063 | - |
| Panel B | - | 1.78 | 5.19e-03 | 0.25 | - |
| | Representative industries | | | | |
| Industry 2421 | Sawmills and planing mill, general | 1.47 | -7.51e-03 | 0.400 | 0.4642 |
| Industry 2869 | Industrial organic chemicals | 1.40 | -1.00e-02 | 0.23 | 0.5286 |
| Industry 2911 | Petroleum mining | 1.00 | -8.02e-04 | 0.40 | 0.6504 |
| Industry 3084 | Plastic pipes | 1.17 | -8.03e-04 | 0.31 | 0.4235 |
| Industry 3315 | Steel wiredrawing, steel nails and spikes | 1.12 | 8.76e-03 | 0.57 | 0.3646 |
| Industry 3448 | Fabricated metal buildings and components | 1.43 | -9.49e-04 | 0.23 | 0.3948 |
| Industry 3519 | Internal combustion engines | 1.01 | -4.82e-03 | 0.39 | 0.4255 |
| Industry 3621 | Motors and generators | 1.12 | 1.06e-02 | 0.34 | 0.3513 |
| Industry 3731 | Ship building and repairing | 1.73 | 1.14e-03 | 0.28 | 0.4072 |
| | High and Low Uncertainty Industries | | | | |
| Industry 3571 | Electronic computers | 7.46 | 2.47e-02 | 0.88 | 0.6222 |
| Industry 2075 | Soybean oil mills | 1.80 | 3.15e-03 | 0.42 | 0.6875 |
| Industry 2879 | Pesticides and agricultural chemicals, not elsewhere classified | 1.11 | 4.40e-03 | 1.01 | 0.5824 |
| Industry 3221 | Glass containers | 1.23 | 8.703-04 | 0.23 | 0.1934 |

*Note*: The model is simulated based on the decision rules for the main model with ambiguity where the agent cannot distinguish perfectly between two processes with persistence $\rho_l = 0.30$ and $\rho_h = 0.85$. The riskfree interest rate is simulated for the full sample of 1947:i-2015:IV but the simulated values for the period 1947:I-1978:IV are discarded as part of the burn-in sample. The distorted means, $\tilde{X}$, and the distorted standard deviations, $SD(\tilde{X})$, are computed according to formulas in Section 5 and are evaluated in percentage terms, as is the risk-free rate. The uncertainty measure is taken from Bloom *et al.* (2012).