Looking at the FTPL through a Unified Macro Model

Kazuyuki Sasakura*

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Abstract

Motivated by disputes over the quantity theory of money and the fiscal theory of the price level (the FTPL), I consider price determination in another macro model with the government and the foreign sector. The model is based on Keynes's *General Theory* but can be used both in the short run and in the long run. It is assumed that prices are determined *by supply and demand in the short run* unlike in the quantity theory and the FTPL. In particular the determination of the price of consumption goods is pursued. Finally the FTPL is examined using the model.

Key words: Price determination, Quantity theory of money, Fiscal theory of the price level, Neoclassical synthesis

JEL classification: E12, E13, E31

1 Introduction

The century before last Wicksell said, "I already had my suspicions ... that, as an alternative to the Quantity Theory, there is no complete and coherent theory of money. If the Quantity Theory is false —or to the extent that it is false— there is so far available only one false theory of money, and no true theory. ... It is no exaggeration to say that even to-day many of the most distinguished economists lack any real, logically worked out theory of money ...." (from the English translation (1936) of Wicksell (1898, p. iii)). Then, as is well-known, he argued that the price rises (falls, and remains unchanged) if the nominal rate of interest is less than (greater than, equal to) the natural rate of interest. Wicksell's challenge to the quantity theory of money had great influence on many contemporary economists.

Keynes was among them and wrote a *Treatise on Money* (1930) replacing the difference between the two rates of interest in Wicksell with that between investment and saving as the determinant of the price level. But he admitted that it had been a failure. Struggling to escape from the quantity theory Keynes took a different approach in his *General Theory* (1936). According to it, the price level is determined by supply and demand in the same way as individual prices. It is noteworthy that in both his books prices respond to market conditions and are flexible. This is contrary to the assumption of price rigidity in both traditional Keynesians who regarded the price to be given and new Keynesians who regard it to be fixed by profit-maximizing firms.

In fact there were the pros and cons about the relevance of the quantity theory of money even among the greatest economists. For example, Ricardo, Walras, Marshall, and Fisher

*Faculty of Political Science and Economics, Waseda University, Japan. E-mail: sasakura@waseda.jp

supported it, whereas Adam Smith and Schumpeter as well as Wicksell and Keynes did not believe it. After such an indefinite period, however, the quantity theory held sway over macroeconomics. Needless to say, it is Friedman (1956) as a monetarist who revived it and it is Lucas (1972) who inherited it in the rational expectations framework. The long-lasting belief in the quantity theory may come from its theoretical simplicity and overwhelming evidence for the positive relationship between the rate of change in the money supply and the rate of inflation as Lucas (2007) stuck to it. The quantity theory continues to be a theoretical core at least in neoclassical theory. Solow's (1956) neoclassical growth model, a universal basis of modern macroeconomics, also relies on it.

Nevertheless, not everybody has been convinced. Recently a new theory called the fiscal theory of the price level (the FTPL, hereafter) has been advocated by leading macroeconomists including Christiano and Fitzgerald (2000), Cochrane (1998, 2005), Leeper (1991), Leeper and Walker (2013), Sims (1994, 2013), and Woodford (1994, 1995, 1998).[1] The FTPL claims that the price level is so determined that the real value of government debt (i.e., nominal government debt divided by the price level) may become equal to the present discounted value of government primary surpluses. This equality between the two values is considered to be the equilibrium condition for the determination of the price level. Cochrane (1998, p. 328) went so far as to say that it "will determine the price level no matter what the rest of the economy looks like, so we don't have to spell it out." Then the FTPL argues that the government budget constraint is satisfied *as a result of the determination of the equilibrium price level.*[2]

Models of the FTPL are usually neoclassical intertemporal general equilibrium models which always involve the government sector collecting taxes and issuing bonds. In addition, they always have "frictions" in them such as cash in advance, money in utility, and transactions costs. Due to such frictions, the papers on the FTPL are very technical and even "esoteric" as Sims (2013, p. 564) said. There is also quite a few criticism of it, e.g., from Bassetto (2002), Bohn (1998), Buiter (2002), Carlstrom and Fuerst (2000), and McCallum (2001). Notably Buiter (2002) set up his own environment and concluded that the FTPL is fatally flawed because it confuses a budget constraint, which must always be satisfied, and an equilibrium condition, which is required to hold by a theory.[3] A problem is that there is no decisive statistical method of distinguishing between the quantity theory and the FTPL.[4] Then, what should we learn from these disputes (though mainly among neoclassicists)?

In my opinion the quantity theory of money has not been dethroned by the FTPL. At least it is not easy to imagine transactions of more than one kind of goods without using money as a medium of exchange. On the other hand, the FTPL gave an interesting viewpoint to modern monetary theory by emphasizing the importance of fiscal policy in the determination of the price level. It is certainly meaningful to ask a question of what will become of the quantity theory if the government sector is incorporated explicitly. Then, how about the foreign sector? It would be more convincing if the foreign sector can be introduced as well

[1] It is interesting to notice that Woodford (1994) starts with the description of modern monetary theory very similar to Wicksell (1898) nearly a century ago. The FTPL dates back to Sargent and Wallace (1981) and was ingeniously formulated by Aiyagari and Gertler (1985).

[2] Except academic journals, the FTPL appears in Ljungqvist and Sargent's (2012) advanced textbook and a dictionary of economics (Basseto (2008)), whereas neither Romer's (2012) advanced textbook nor De Vroey's (2016) history of macroeconomics includes it.

[3] Later Woodford (2003) seems to avoid mentioning the FTPL despite the title of the book.

[4] Canzoneri et al. (2011, p. 964) concludes that "in the end, plausibility like beauty may be in the eye of the beholder."

because the price level usually refers to the consumer price index which is made up of the prices of domestic and foreign consumption goods. Thus, the price determination should be studied within an inclusive system of a macroeconomy, not focusing on its specific aspect. That is what I have learned from such disputes.

In sum the issue raised by Wicksell and Keynes long ago is not solved completely even today. Then, the main purpose of this paper is to consider price determination in another macro model with the government sector and the foreign sector. The model can be used both in the short run and in the long run. In such a sense it is a unified macro model. It is assumed that prices are determined *by supply and demand in the short run* unlike in the quantity theory and the FTPL. In particular the determination of the price of *consumption goods* is pursued as suggested above. In addition the FTPL is examined using the model.

This paper is organized as follows. Section 2 analyzes the short-run equilibrium state by constructing the short-run model also called the Keynes model in which prices and outputs are determined through the adjustment in the goods markets as in the *General Theory*. Section 3 analyzes the long-run equilibrium state by turning the short-run model into the long-run model also called the Solow model in which outputs are determined by factor endowments. In the long run the economy finally reaches a steady state. Section 4 looks into the FTPL using the unified model also called the Keynes-Solow model. In my view the FTPL is a long-run theory or a neoclassical theory. So it is appropriate to consider it in the long-run equilibrium state. Section 5 concludes that although the FTPL made a contribution in that it turned our attention to the role of government debt, the price level is not determined along the line of the FTPL.

2 The Short-Run Model (The Keynes Model)

2.1 The Structure of the Model

It would be helpful to explain what the model looks like. It is based considerably on Keynes (1936), not on Keynes (1930) or on traditional Keynesian economics with a sticky price.[5] Prices are flexible and determined by supply and demand as in microeconomics. The principle of effective demand still holds in the sense that investment determines saving as in traditional Keynesian economics. On the other hand, the liquidity preference theory is not relied on since money is used only as a medium of exchange.

There are five sectors in this model, that is, the household sector, the production sector, the government sector, the central bank, and the foreign sector. The production sector consists of the investment-goods sector and the consumption-goods sector which produce respectively investment goods and consumption goods using labor and capital stock. Thus it is a two-sector model.

Time is discrete proceeding, e.g., from period $t-1$ to period t as in Figure 1.[6] Each period is divided into three sub-periods. The first sub-period is that of production, distribution, and expenditure as explained in usual macroeconomics. Value added generated during the production process is distributed to the household sector as wages, interests, and dividends and to the government sector as indirect taxes. Corresponding to the production sector, the goods market consists of the investment-goods market and the consumption-goods market.

[5] The interpretation of Keynes (1936) in this context is due to my own investigations of it in Sasakura (2009, 2016).

[6] Symbols therein will be explained in detail in Section 2.2.

The household sector and the government sector use their incomes to consume or save, and the consumption-goods market clears through the adjustment of price and output. Newly produced investment goods are all bought and added to the existing capital stock.

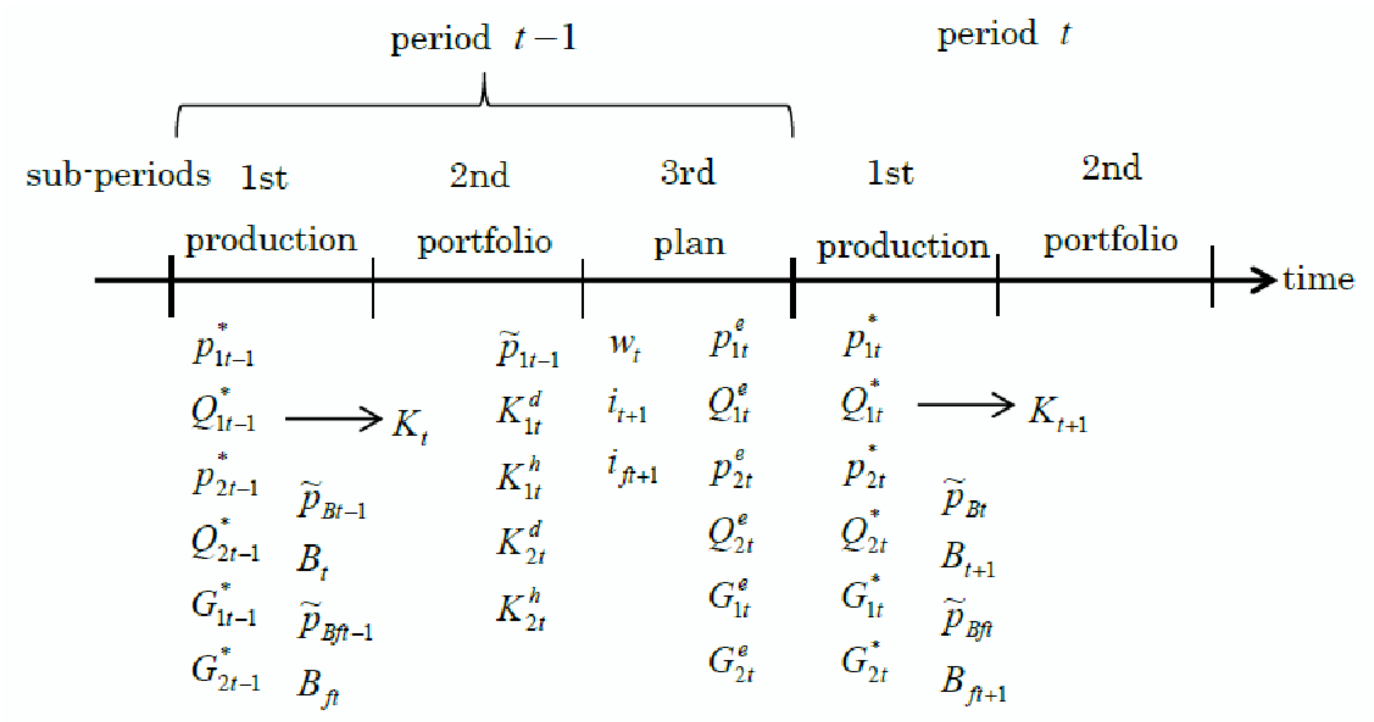

Figure 1. The Time Structure of the Model

The second sub-period is that of the portfolio selection. At the beginning of the second sub-period there are three types of assets, that is, capital stock, government bonds, and foreign bonds which represent foreign assets as a whole. The rates of interest on government bonds and foreign bonds are already fixed. As for capital stock, the household sector can lend it to both the investment-goods sector and the consumption-goods sector. In each case, the household can choose the rate of return between the two kinds. One is the same as the rate of interest on government bonds, while the other is the expected rate of return on capital stock which is not known in the second sub-period.

The third sub-period is that of the plan for the first sub-period of the next period. There already exists capital stock in each production sector as a result of the portfolio selection during the previous sub-period. At the beginning of the third sub-period the nominal rate of wages is fixed through the negotiation between the production sector and the household sector. Given capital stock and the nominal rate of wages, each production sector calculates the amount of labor which maximizes the expected rate of return on capital stock. If the central bank promises to supply money required by the production sector, then the plan is realized in the first sub-period of the next period as explained above with respect to the current period. These processes repeat themselves over and over again.

2.2 The Principle of Effective Demand

Suppose that the economy is at the end of the third sub-period of period $t-1$, and confirm the following identity of income and the values of outputs for the first sub-period of period t:

$$Y_t^e = p_{1t}^e Q_{1t}^e + p_{2t}^e Q_{2t}^e + F_t^e, \tag{1}$$

where all terms are denominated in domestic currency and a superscript e means an expected or planned value which is calculated at the end of the third sub-period.[7] Y_t^e is nominal gross national income (GNI). p_{1t}^e and Q_{1t}^e represent the price and output of domestic investment goods, while p_{2t}^e and Q_{2t}^e represent the price and output of domestic consumption goods. The investment-goods sector and the consumption-goods sector generate gross value added respectively by $p_{1t}^e Q_{1t}^e$ and $p_{2t}^e Q_{2t}^e$. By definition nominal gross national product (GDP) is the sum of the two. By definition again GNI on the left-hand is the sum of GDP and net income F_t^e from the rest of the world on the right-hand side.

Nominal gross private saving S_t^e is defined as

$$S_t^e = Y_t^e - T_t^e - C_t^e, \tag{2}$$

where T_t and C_t^e are total taxes and private consumption, respectively. Using (2), (1) can be expressed as

$$Y_t^e = C_t^e + G_{2t}^e + S_t^e + (T_t^e - G_{2t}^e).$$

This implies that GNI is used for either private consumption or private saving by the household sector, or for either government consumption G_{2t}^e or government saving $T_t^e - G_{2t}^e$ by the government sector. In other words, GNI is used for either national consumption $C_t^e + G_{2t}^e$ or national saving $S_t^e + (T_t^e - G_{2t}^e)$. Since (1) and (2) are definitions, they hold true always, that is, both in the short run and in the long run.

Equilibrium in the investment-goods market is written as

$$p_{1t}^e Q_{1t}^e + IM_{1t}^e = I_t^e + G_{1t}^e + EX_{1t}^e,$$

where IM_{1t}^e, I_t^e, G_{1t}^e, and EX_{1t}^e are respectively imports of foreign investment goods, part of private saving which goes to the purchase of investment goods, government investment, and exports of domestic investment goods. Note that I_t^e is *not* what is usually called "investment" by firms but the expenditure on investment goods, domestic or foreign, by the household sector. As is apparent, the above equation shows the equality of the total amount of the supply of investment goods on the left-hand side and the total amount of the demand for investment goods on the right-hand side.

Equilibrium in the consumption-goods market is written as

$$p_{2t}^e Q_{2t}^e + IM_{2t}^e = C_t^e + G_{2t}^e + EX_{2t}^e,$$

where IM_{2t}^e and EX_{2t}^e are respectively imports of foreign consumption goods and exports of domestic consumption goods. The above equation shows the equality of the total amount of the supply of consumption goods on the left-hand side and the total amount of the demand for consumption goods on the right-hand side.

[7] As will be seen below, a superscript $*$ represents a realized value in the short run, and a superscript $**$ represents a realized value in the long run.

Equilibrium in each market can be rewritten as

$$
\begin{aligned}
p^e_{1t}Q^e_{1t} &= I^e_t + G^e_{1t} + NX^e_{1t}, && (3)\\
p^e_{2t}Q^e_{2t} &= C^e_t + G^e_{2t} + NX^e_{2t}, && (4)
\end{aligned}
$$

where NX^e_{1t} is net exports of investment goods defined as $NX^e_{1t} = EX^e_{1t} - IM^e_{1t}$, and NX^e_{2t} is net exports of consumption goods defined as $NX^e_{2t} = EX^e_{2t} - IM^e_{2t}$.[8] Substituting (3) and (4) into (1) yields

$$Y^e_t = C^e_t + I^e_t + (G^e_{1t} + G^e_{2t}) + CA^e_t,$$

where CA^e_t is current account defined as $CA^e_t = NX^e_{1t} + NX^e_{2t} + F^e_t$. The above equation seems to be a familiar equation for "equilibrium in the goods market," but it is just the result of equilibria in the two goods markets (3) and (4).

When the consumption-goods market is in equilibrium as (4) shows, gross private saving (2) can be expressed as

$$
\begin{aligned}
S^e_t &= p^e_{1t}Q^e_{1t} + G^e_{2t} + NX^e_{2t} - T^e_t + F^e_t\\
&= I^e_t + S^e_{Bt} + S^e_{Bft}, && (5)
\end{aligned}
$$

where three terms I^e_t, S^e_{Bt}, and S^e_{Bft} represent respectively parts of the saving used for purchasing investment goods, government bonds, and foreign bonds. I^e_t is related to the capital accumulation equation

$$K^e_{t+1} = (1-\delta)K_t + Q^e_{1t} - \frac{EX^e_{1t}}{p^e_{1t}} + \frac{IM^e_{1t}}{e^e_t p^e_{f1t}}, \qquad (6)$$

where K^e_{t+1} and K_t are respectively capital stock for production of period $t+1$ and capital stock for production of period t. δ is the capital depreciation rate ($0 \leq \delta \leq 1$). (6) may be regarded just as a physical causality, but it does make economic sense only if newly added capital stock is bought and held by the household and government sectors as follows:

$$
\begin{aligned}
& p^e_{1t}Q^e_{1t} - EX^e_{1t} + IM^e_{1t}\\
=\ & I^e_t + G^e_{1t}\\
=\ & p^e_{1t}K^e_{t+1} - p^e_{1t}(1-\delta)K_t - p^e_{1t}\cdot TG^e_{1t}, && (7)
\end{aligned}
$$

where TG^e_{1t} represents trading gains of investment goods defined as $\left[1 - \left(\frac{p^e_{1t}}{e^e_t p^e_{f1t}}\right)^{-1}\right]\frac{IM^e_{1t}}{e^e_t p^e_{f1t}}$ with p^e_{f1t} as the (expected) price of foreign investment goods in foreign currency. The first and second lines of (7) is none other than equilibrium in the investment-goods market (3). The first and third lines lead to the capital accumulation equation (6). As is seen from the second and third lines, capital stock increases by more than (less than) $(I^e_t + G^e_{1t})/p^e_{1t}$ if $TG^e_{1t} > (<) 0$.[9]

S^e_{Bt} is related to the "government-bond accumulation equation" or in usual terms the government budget constraint

$$G^e_{1t} + G^e_{2t} + (1+i_t)\tilde{p}_{Bt-1}B_t = IT^e_t + DT^e_t + \tilde{p}_{Bt}B^e_{t+1}. \qquad (8)$$

[8] In a sense the equilibrium conditions (3) and (4) are truism. But it is hard to find economists who resort to them.

[9] As will be seen, trading gains vanish in the long run.

The left-hand side of (8) represents the total amount the government sector must pay in the first sub-period of period t. $G^e_{1t} + G^e_{2t}$ is total government expenditure. $(1 + i_t)\tilde{p}_{Bt-1}B_t$ is the sum of principal and interest of government bonds B_t already issued at the price $\tilde{p}_{Bt-1}$ in the first sub-period of period $t - 1$ (See Figure 1). It is assumed that the nominal rate i_t of interest and the bond price $\tilde{p}_{Bt-1}$ are set respectively by the central bank and by the government in the third sub-period of period $t - 2$.

The right-hand side of (8) shows how the government sector covers the amount on the left-hand side. IT^e_t and DT^e_t are indirect taxes and direct taxes, respectively. Indirect taxes are specified as

$$IT^e_t = \mu(p^e_{1t}Q^e_{1t} + p^e_{2t}Q^e_{2t}), 0 \leq \mu < 1, \tag{9}$$

where μ is the rate of indirect taxes which is regarded to be a parameter. The government sector satisfies (8) by issuing bonds B^e_{t+1} at the price $\tilde{p}_{Bt}$. As a result, the sum of principal and interest paid in the first sub-period of period $t + 1$ amounts to $(1 + i_{t+1})\tilde{p}_{Bt}B^e_{t+1}$.

As in the capital accumulation equation, the government-bond accumulation equation (8) also makes economic sense only if newly issued government bonds are bought by the household sector as follows:

$$\begin{aligned} BD^e_t &= G^e_{1t} + G^e_{2t} - T^e_t \\ &= S^e_{Bt} \\ &= \tilde{p}_{Bt}B^e_{t+1} - \tilde{p}_{Bt-1}B_t. \end{aligned} \tag{10}$$

BD^e_t in the first line is government budget deficits defined as $G^e_{1t} + G^e_{2t} - T^e_t$, where T^e_t is total taxes defined as $IT^e_t + DT^e_t - i_t\tilde{p}_{Bt-1}B_t$. T^e_t in (10) is the same as that in (2). The first and third lines lead to the government budget constraint (8). Then, (10) implies that the government sector makes up the deficits BD^e_t by borrowing S^e_{Bt} from the household sector with the result that the household sector holds government bonds the nominal value of which amounts to $\tilde{p}_{Bt}B^e_{t+1}$.

S^e_{Bft} is related to the "foreign-bond accumulation equation"

$$EX^e_{1t} + EX^e_{2t} + e^e_t(1 + i_{ft})\tilde{p}_{Bft-1}B_{ft} = IM^e_{1t} + IM^e_{2t} + e^e_t\tilde{p}_{Bft}B^e_{ft+1}. \tag{11}$$

As is obvious, (11) resembles (8). So (11) may be called the "foreign budget constraint," and interpreted in a similar way. That is, the left-hand side of (11) represents the total amount the foreign sector must pay in the first sub-period of period t. $EX^e_{1t} + EX^e_{2t}$ is the amount the economy obtains from the foreign sector because the foreign sector *imports* the investment and consumption goods from the economy. $e^e_t(1 + i_{ft})\tilde{p}_{Bft-1}B_{ft}$ is the sum of principal and interest of foreign bonds B_{ft} already issued at the price $\tilde{p}_{Bft-1}$ in the first sub-period of period $t - 1$ (See again Figure 1). It is denoted in terms of currency of the economy since it is multiplied by the (expected) nominal rate of exchange e^e_t. i_{ft} is the already fixed nominal rate of interest on the foreign bonds.

The right-hand side of (11) shows how the foreign sector covers the amount on the left-hand side. $IM^e_{1t} + IM^e_{2t}$ is the amount the foreign sector obtains from the economy in the first sub-period of period t because the foreign sector *exports* the investment and consumption goods to the economy. The foreign sector satisfies (11) by issuing foreign bonds B^e_{ft+1} at the price $\tilde{p}_{Bft}$. As a result, the sum of principal and interest paid in the first sub-period of period $t + 1$ amounts to $e^e_{t+1}(1 + i_{ft+1})\tilde{p}_{Bft}B^e_{ft+1}$.

The foreign-bond accumulation equation (11) makes economic sense too only if newly issued foreign bonds are bought by the household sector as follows:

$$\begin{aligned} CA_t^e &= EX_{1t}^e + EX_{2t}^e - (IM_{1t}^e + IM_{2t}^e) + F_t^e \\ &= S_{Bft}^e \\ &= e_t^e \tilde{p}_{Bft} B_{ft+1}^e - e_t^e \tilde{p}_{Bft-1} B_{ft}. \end{aligned} \tag{12}$$

CA_t^e in the first line is current account already defined and $F_t^e = e_t^e i_{ft} \tilde{p}_{Bft-1} B_{ft}$. F_t^e in (12) is the same as that in (1). As is easily seen, the first and third lines lead to the foreign budget constraint (11). Here it is convenient to think that the economy receives CA_t^e in the form of foreign currency. Then, (12) implies that the household sector buys newly issued foreign bonds by the same amount of foreign currency with the result that the household sector holds foreign bonds the nominal value of which amounts to $e_t^e \tilde{p}_{Bft} B_{ft+1}^e$.

Note that the sum of the first lines of (7), (10), and (12) is equal to the first line of (5). Thus, the following lemma obtains:

Lemma: Suppose that the consumption-goods market is in equilibrium. Then, the realization of arbitrary two among (7), (10), and (12) implies that of the rest of the three.

If (7) does not hold, no investment goods are produced in the economy. And if (12) does not hold, foreign currency obtained remains idle. Thus, it is reasonable economically to assume as follows:

Assumption 1: Both (7) and (12) hold in each period.

Assumption 1 implies that both equilibrium in the investment-goods market (3) and the foreign budget constraint (11) are realized in each period. Then, the above lemma leads to the following proposition:

Proposition 1: Suppose that the consumption-goods market is in equilibrium. Then, under Assumption 1 the government budget constraint (8) is always realized.

The relationships among many variables derived above correspond pretty well to the system of national accounts according to which key macro data are collected, compiled, and used for macro analysis in each country. In this sense they take a comprehensive overview of a macroeconomy without serious omissions. However, no theory of determination of national income is included there. In other words there is no distinction between endogenous and exogenous variables. Then, according to Keynes (1936), let us regard Y_t^e, C_t^e, and $p_{2t}^e Q_{2t}^e$ as endogenous and the other variables as exogenous.

But the values of these endogenous variables are still indeterminate since there are only two equations (1) and (4) to the three unknowns. Needless to say, it is a consumption function that is needed to determine them. So I adopt a simple one as follows:

$$C_t^e + G_{2t}^e = cY_t^e, \; 0 < c < 1, \tag{13}$$

where c is the ratio of national consumption to GNI. The household sector chooses the value of c which is assumed to be constant over time.[10] Substituting (13) into (4) and taking (1) into consideration yields equilibrium nominal output of domestic consumption goods with $p_{1t}^e Q_{1t}^e$, F_t^e, and NX_{2t}^e as given:

$$p_{2t}^e Q_{2t}^e = \frac{c(p_{1t}^e Q_{1t}^e + F_t^e) + NX_{2t}^e}{1-c}. \tag{14}$$

At the same time equilibrium GNI is calculated as

$$Y_t^e = \frac{1}{1-c}\left(p_{1t}^e Q_{1t}^e + F_t^e + NX_{2t}^e\right). \tag{15}$$

If the economy is a closed one, only nominal output $p_{1t}^e Q_{1t}^e$ of investment goods and the propensity to consume c determine national income.[11]

The theoretical developments so far are of the traditional Keynesian type. Particularly it should be noticed that the principle of effective demand obtains since "investment" $p_{1t}^e Q_{1t}^e$ determines "saving" S_t^e as the first line of (5) shows. That is why the short-run model is called the Keynes model. But there are two differences between the Keynes model here and the traditional Keynesian model. First, notice that government investment G_{1t}^e does not appear on the right-hand side of (15). This fact can be stated as follows:

Proposition 2: In the Keynes model government investment has no impact on equilibrium national income.

Second, prices are not rigid but flexible in the Keynes model. The next subsection elaborates on the production sector under such circumstances.[12]

[10]The constancy of c does not imply that the value of c is unique. It can vary depending on the objective of the household sector, e.g., as shown in Appendices E and F.

[11]In equilibrium the ratio of national consumption to national saving is expressed as

$$\begin{aligned} \frac{C_t^e + G_{2t}^e}{S_t^e + (T_t^e - G_{2t}^e)} &= \frac{c}{1-c} \\ &= \frac{p_{2t}^e Q_{2t}^e - NX_{2t}^e}{p_{1t}^e Q_{1t}^e + F_t^e + NX_{2t}^e}, \end{aligned}$$

which means that the household sector divides GNI into national consumption and national gross saving in the ratio of c to $1-c$ with $p_{1t}^e Q_{1t}^e + F_t^e + NX_{2t}^e$ as given. In the case of a closed economy the ratio is simply $p_{2t}^e Q_{2t}^e$ to $p_{1t}^e Q_{1t}^e$.

[12]Substituting (13) into (4) and ignoring the foreign sector gives $p_{2t}^e Q_{2t}^e = cY_t^e$. This corresponds to the basic equation Lindahl used to explain the factors determining changes in the price level. On the basis of the form $p_{2t}^e = \frac{cY_t^e}{Q_{2t}^e}$, Lindahl (1939, p. 146) argued that "a change in the price level for consumed goods (and services) presupposes that the nominal income allotted to consumption has been altered relatively to the quantity of consumption goods." His reasoning on price changes applies to the Keynes model too.

2.3 The Production Sector

At the end of each third sub-period the price of investment goods of the first sub-period of the next period is expected as shown in Figure 1. Such expected prices may vary period by period. Moreover, corresponding prices of consumption goods may also change so that the consumption-goods market can clear. In order to understand how prices change it is necessary to introduce production functions which are usually hidden in traditional Keynesian economics.

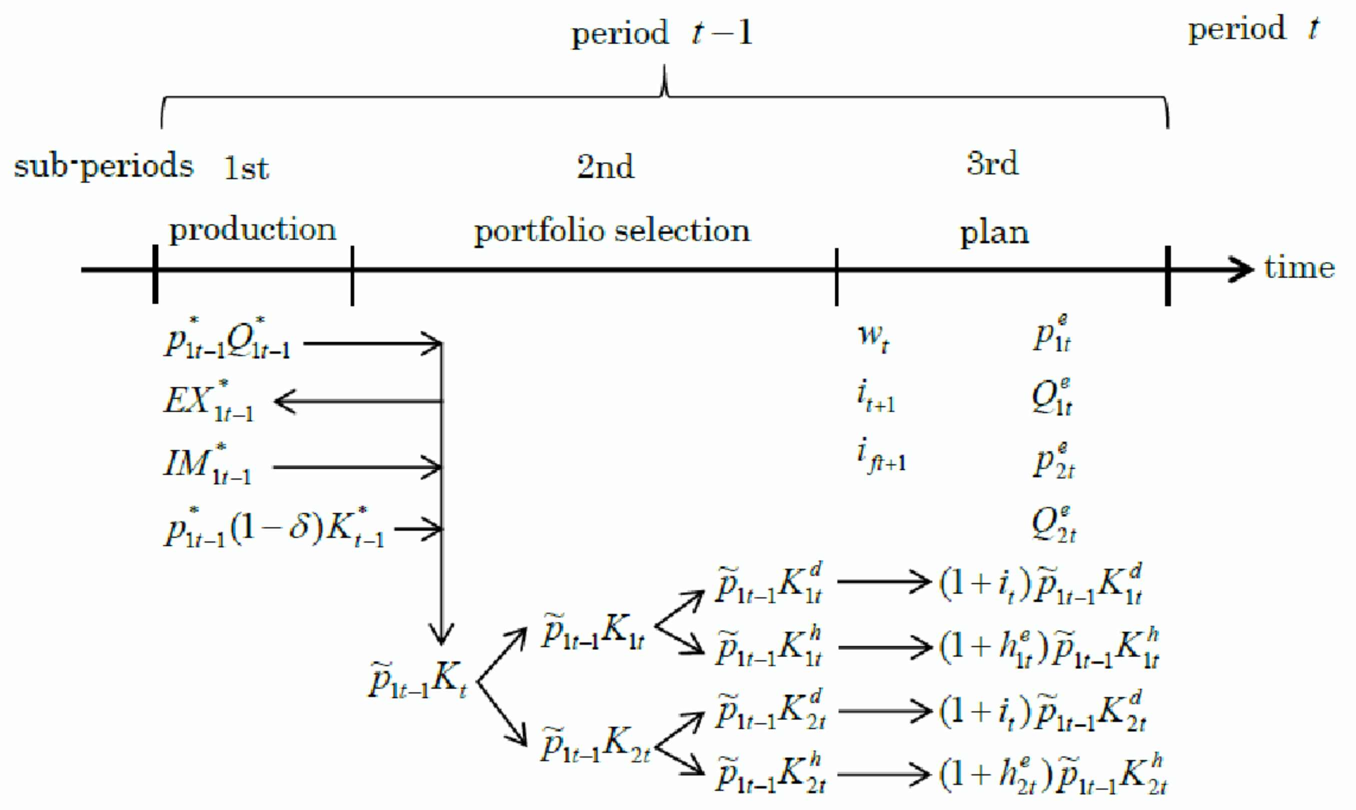

Figure 2. Four Ways to Hold Capital Stock as an Asset

The technology of the investment-goods sector is given by the Cobb-Douglas production function:

$$Q_{1t} = K_{1t}^{\alpha}(A_t N_{1t})^{1-\alpha}, \quad K_{1t} = K_{1t}^d + K_{1t}^h, \; 0 < \alpha < 1, \tag{16}$$

$$A_t = (1+g)A_{t-1}, \tag{17}$$

where Q_{1t}, K_{1t}, N_{1t}, and A_t are respectively output, capital stock, labor used, and the effectiveness of labor of the investment-goods sector in the first sub-period of period t. The effectiveness of labor or "knowledge" is assumed to grow at an exogenous rate g as in (17).[13] As Figure 2 shows, K_{1t} is held as either K_{1t}^d or K_{1t}^h as a result of the portfolio selection in

[13] g must be greater than -1. Its admissible value is specified in Section 3.2.

the second sub-period of period $t-1$. During the portfolio selection the price $\tilde{p}_{1t-1}$ of K_{1t} is determined. In addition the holders of K^d_{1t} are promised to receive the fixed sum of principal and interest in the first sub-period of period t, whereas the holders of K^h_{1t} don't know what the return on it will be.

As said in Section 2.1, the nominal rate w_t of wage is determined at the beginning of the third sub-period of period $t-1$. After that the price of investment goods of the first sub-period of period t is expected. Then, the expected rate h^e_{1t} of return on capital stock K^h_{1t} can be calculated from the following definition:

$$\begin{aligned} &p^e_{1t}Q^e_{1t} + p^e_{1t}(1-\delta)K_{1t} \\ &= w_t N^e_{1t} + (1+i_t)\tilde{p}_{1t-1}K^d_{1t} + (1+h^e_{1t})\tilde{p}_{1t-1}K^h_{1t} + \mu p^e_{1t}Q^e_{1t}. \end{aligned}$$

Rewriting it in terms of flow yields

$$p^e_{1t}Q^e_{1t} = w_t N^e_{1t} + i_t\tilde{p}_{1t-1}K^d_{1t} + h^e_{1t}\tilde{p}_{1t-1}K^h_{1t} + \mu p^e_{1t}Q^e_{1t} + p^e_{1t}(\delta - \pi^e_t)K_{1t}, \tag{18}$$

where μ is the rate of indirect taxes as in (9), and $\pi^e_t = 1 - (\tilde{p}_{1t-1}/p^e_{1t})$. Clearly $\pi^e_t < 1$. For convenience' sake let us call π^e_t and $\delta - \pi^e_t$ respectively the "expected inflation rate" and the "inflation-adjusted depreciation rate."[14] (18) means that value added $p^e_{1t}Q_{1t}$ generated by the investment-goods sector is distributed as labor income, capital income, indirect taxes, or "inflation-adjusted capital depreciation."

The mission of the investment-goods sector is to maximize the expected rate h^e_{1t} of return in (18) subject to the production technology (16) and (17). Solving (18) for h^e_{1t} gives

$$h^e_{1t} = \frac{(1-\mu)p^e_{1t}Q^e_{1t} - w_t N^e_{1t} - i_t\tilde{p}_{1t-1}K^d_{1t} - p^e_{1t}(\delta - \pi^e_t)K_{1t}}{\tilde{p}_{1t-1}K^h_{1t}}. \tag{19}$$

Since the right-hand side of (19) is a function of N^e_{1t} alone, the investment-goods sector has only to find the level of labor, N^e_{1t}, which maximizes h^e_{1t}. Substituting (16) into (19) and differentiating with respect to N^e_{1t} yields

$$\frac{dh^e_{1t}}{dN^e_{1t}} = \frac{(1-\mu)p^e_{1t}(1-\alpha)A_t^{1-\alpha}(N^e_{1t})^{-\alpha}K^\alpha_{1t} - w_t}{\tilde{p}_{1t-1}K^h_{1t}}.$$

Then N^e_{1t} can easily be obtained by solving $dh^e_{1t}/dN^e_{1t} = 0$ and $d^2h^e_{1t}/d(N^e_{1t})^2 < 0$ as follows:

$$N^e_{1t} = \left[(1-\alpha)A_t^{1-\alpha}\frac{(1-\mu)p^e_{1t}}{w_t}\right]^{\frac{1}{\alpha}} K_{1t}. \tag{20}$$

[14] An unfamiliar term π^e_t can be written as

$$\pi^e_t = \frac{\frac{p^e_{1t}-\tilde{p}_{1t-1}}{\tilde{p}_{1t-1}}}{1 + \frac{p^e_{1t}-\tilde{p}_{1t-1}}{\tilde{p}_{1t-1}}}.$$

Thus, π^e_t is approximately equal to a kind of expected inflation rate $\frac{p^e_{1t}-\tilde{p}_{1t-1}}{\tilde{p}_{1t-1}}$ when it is close to zero. Another unfamiliar term $\delta - \pi^e_t$ can be understood by using the following expression:

$$p^e_{1t}(\delta - \pi^e_t)K_{1t} = \tilde{p}_{1t-1}\delta K_{1t} - (p^e_{1t} - \tilde{p}_{1t-1})(1-\delta)K_{1t}.$$

$\tilde{p}_{1t-1}\delta K_{1t}$ on the right-hand side corresponds to what is usually called capital depreciation, i.e., the money necessary to restore depreciated capital to the original *nominal* value $\tilde{p}_{1t-1}K_{1t}$. The above expression states more correctly that such money can be decreased by $(p^e_{1t} - \tilde{p}_{1t-1})(1-\delta)K_{1t}$ when the price of investment goods rises from $\tilde{p}_{1t-1}$ to p^e_{1t} but it must be increased by $(\tilde{p}_{1t-1} - p^e_{1t})(1-\delta)K_{1t}$ when the price of investment goods falls from $\tilde{p}_{1t-1}$ to p^e_{1t}.

For N^e_{1t} in (20) the output of investment-goods is calculated as

$$\begin{aligned} Q^e_{1t} &= K^{\alpha}_{1t}(A_t N^e_{1t})^{1-\alpha} \\ &= \left[(1-\alpha)A_t \frac{(1-\mu)p^e_{1t}}{w_t}\right]^{\frac{1-\alpha}{\alpha}} K_{1t}. \end{aligned} \tag{21}$$

(21) is a supply curve of investment goods as a function of a price p^e_{1t}. In Figure 3 is drawn such a supply curve $Q^S_{1t}(= Q^e_{1t})$.[15] Once p^e_{1t} is set, optimal output Q^e_{1t} of investment goods can be located through the supply curve as the arrows show. The area of the rectangle formed by the two arrows and the two axes is equal to "investment (plan)" $p^e_{1t}Q^e_{1t}$ which is the driving force of the Keynes model as (14) and (15) show.

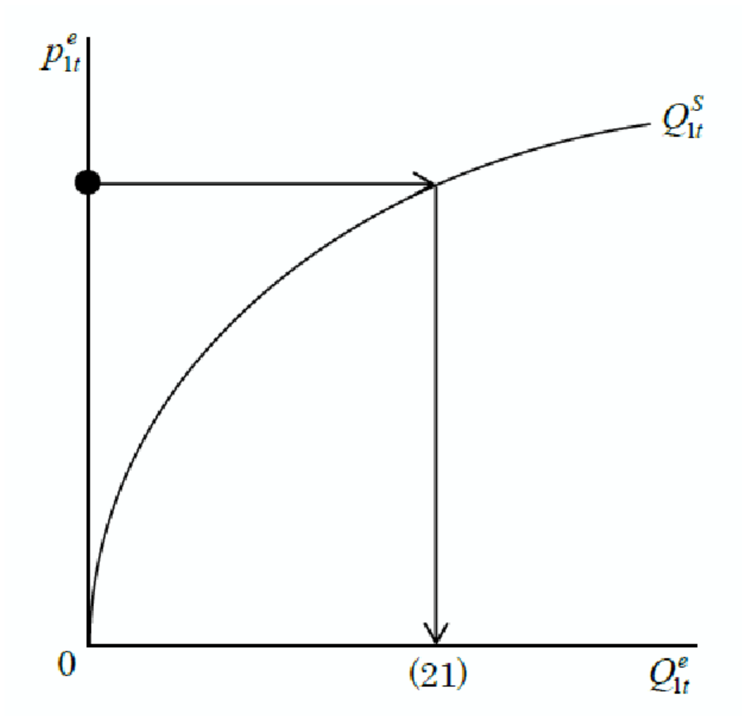

Figure 3. Supply Curve of Investment Goods

Now, let MPL_{1t} be the marginal productivity of labor in period t. Since $MPL_{1t} \equiv \partial Q_{1t}/\partial N_{1t}$, the usual profit maximization condition can be written as

$$MPL^e_{1t} = (1-\alpha)A^{1-\alpha}_t (N^e_{1t})^{-\alpha} K^{\alpha}_{1t} = \frac{w_t}{(1-\mu)p^e_{1t}}, \tag{22}$$

which is equivalent to (20). It follows that the maximization of h^e_{1t} is equivalent to the usual profit maximization. Finally, the marginal productivity of capital in period t, MPK_{1t}, is

$$MPK^e_{1t} = \alpha K^{\alpha-1}_{1t}(A_t N^e_{1t})^{1-\alpha}. \tag{23}$$

[15] For the derivation of it, see Appendix A.

Similar explanations apply to the consumption-goods sector, too. The production function of the consumption-goods sector is given by

$$Q_{2t} = K_{2t}^{\alpha}(A_t N_{2t})^{1-\alpha},\ K_{2t} = K_{2t}^{d} + K_{2t}^{h},\ 0 < \alpha < 1, \tag{24}$$

where K_{2t} and N_{2t} are respectively capital and labor of the consumption-goods sector in the first sub-period of period t. K_{2t} is held as either K_{2t}^{d} or K_{2t}^{h} as a result of the portfolio selection in the second sub-period of period $t-1$. K_{2t}^{d} and K_{2t}^{h} in the consumption-goods sector correspond respectively to K_{1t}^{d} and K_{1t}^{h} in the investment-goods sector (See again Figure 2).

The expected rate h_{2t}^{e} of return on K_{2t}^{h} is calculated from the following definition:

$$\begin{aligned} & p_{2t}^{e} Q_{2t}^{e} + p_{1t}^{e}(1-\delta)K_{2t} \\ & = w_t N_{2t}^{e} + (1+i_t)\tilde{p}_{1t-1} K_{2t}^{d} + (1+h_{2t}^{e})\tilde{p}_{1t-1} K_{2t}^{h} + \mu p_{2t}^{e} Q_{2t}^{e}. \end{aligned}$$

Rewriting it in terms of flow gives

$$p_{2t}^{e} Q_{2t}^{e} = w_t N_{2t}^{e} + i_t \tilde{p}_{1t-1} K_{2t}^{d} + h_{2t}^{e} \tilde{p}_{1t-1} K_{2t}^{h} + \mu p_{2t}^{e} Q_{2t}^{e} + p_{1t}^{e}(\delta - \pi_t^{e})K_{2t}. \tag{25}$$

The mission of the consumption-goods sector is to maximize the expected rate h_{2t}^{e} of return in (25) subject to the production technology (24) and (17). Solving (25) for h_{2t}^{e} yields

$$h_{2t}^{e} = \frac{(1-\mu)p_{2t}^{e} Q_{2t}^{e} - w_t N_{2t}^{e} - i_t \tilde{p}_{1t-1} K_{2t}^{d} - p_{1t}^{e}(\delta - \pi_t^{e})K_{2t}}{\tilde{p}_{1t-1} K_{2t}^{h}}. \tag{26}$$

Substituting (24) into (26) and differentiating with respect to N_{2t}^{e} yields

$$\frac{dh_{2t}^{e}}{dN_{2t}^{e}} = \frac{(1-\mu)p_{2t}^{e}(1-\alpha)A_t^{1-\alpha}(N_{2t}^{e})^{-\alpha} K_{2t}^{\alpha} - w_t}{\tilde{p}_{1t-1} K_{2t}^{h}}.$$

Then N_{2t}^{e} can be obtained by solving $dh_{2t}^{e}/dN_{2t}^{e} = 0$ and $d^2 h_{2t}^{e}/d(N_{2t}^{e})^2 < 0$ as follows:

$$N_{2t}^{e} = \left[(1-\alpha)A_t^{1-\alpha}\frac{(1-\mu)p_{2t}^{e}}{w_t}\right]^{\frac{1}{\alpha}} K_{2t}. \tag{27}$$

The output of consumption goods for N_{2t}^{e} in (27) is calculated as

$$\begin{aligned} Q_{2t}^{e} &= K_{2t}^{\alpha}(A_t N_{2t}^{e})^{1-\alpha} \\ &= \left[(1-\alpha)A_t \frac{(1-\mu)p_{2t}^{e}}{w_t}\right]^{\frac{1-\alpha}{\alpha}} K_{2t}. \end{aligned} \tag{28}$$

(28) is a supply curve of consumption goods as a function of a price p_2^{e}.

Now the following proposition is obtained by substituting (28) into (14):

Proposition 3: The equilibrium price and output of domestic consumption goods can be written as follows:

$$p_{2t}^{e} = \left[\frac{w_t}{(1-\mu)(1-\alpha)A_t}\right]^{1-\alpha}\left[\frac{1}{K_{2t}}\right]^{\alpha}\left[\frac{c(p_{1t}^{e} Q_{1t}^{e} + F_t^{e}) + NX_{2t}^{e}}{1-c}\right]^{\alpha}, \tag{29}$$

$$Q_{2t}^{e} = \left[\frac{(1-\mu)(1-\alpha)A_t}{w_t}\right]^{1-\alpha} K_{2t}^{\alpha}\left[\frac{c(p_{1t}^{e} Q_{1t}^{e} + F_t^{e}) + NX_{2t}^{e}}{1-c}\right]^{1-\alpha}. \tag{30}$$

In Figure 4 are shown a supply curve $Q_{2t}^S (= Q_{2t}^e)$ and a demand curve Q_{2t}^D of consumption goods.[16] It should be emphasized here that (29) is the most important result in this paper because it shows how the *price level* is determined.[17] Proposition 3 leads to the following corollary about (29) at once:

Corollary: If the Keynes model is a closed economy, the equilibrium price p_{2t}^e of domestic consumption goods does not respond to changes in w_t, A_t, or μ, ceteris paribus.

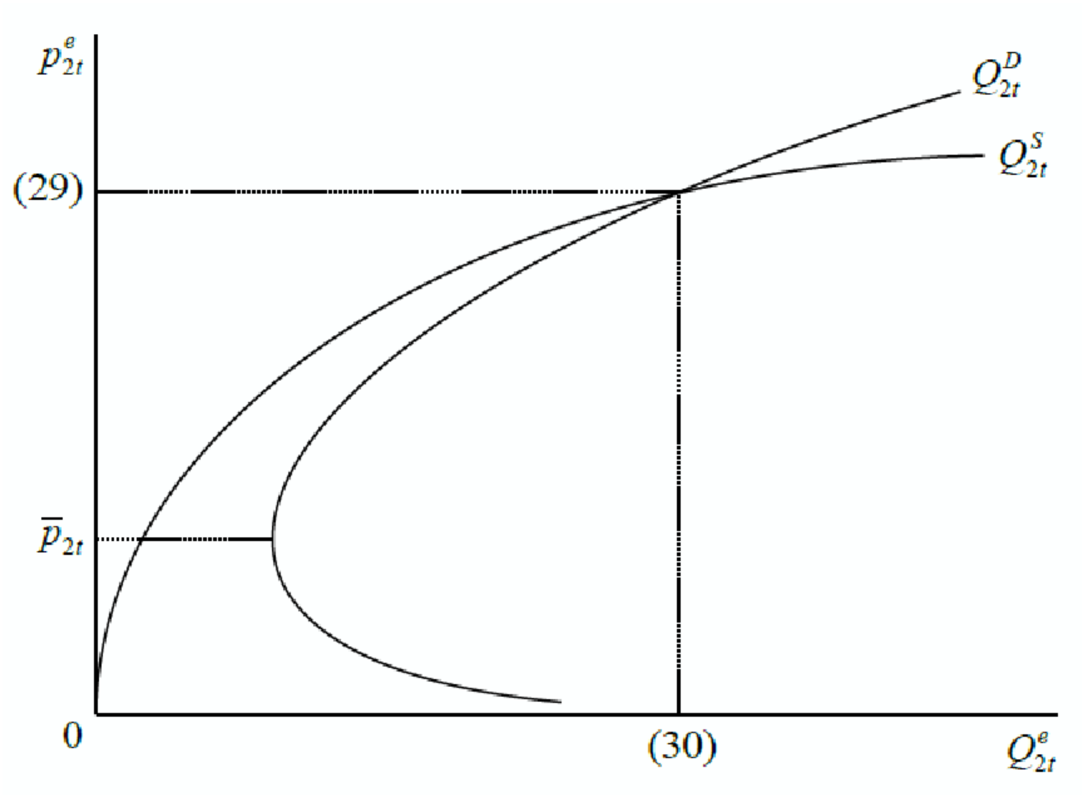

Figure 4. Equilibrium in the Consumption-Goods Market

Finally denote the marginal productivity of labor in period t by MPL_{2t}. Because $MPL_{2t} \equiv \partial Q_{2t}/\partial N_{2t}$, the usual profit maximization condition becomes

$$MPL_{2t}^e = (1-\alpha)A_t^{1-\alpha}(N_{2t}^e)^{-\alpha}K_{2t}^{\alpha} = \frac{w_t}{(1-\mu)p_{2t}^e}, \tag{31}$$

which is equivalent to (27). Therefore, the maximization of h_{2t}^e is equivalent to the usual profit maximization. The marginal productivity of capital in period t, MPK_{2t}, is

$$MPK_{2t}^e = \alpha K_{2t}^{\alpha-1}(A_t N_{2t}^e)^{1-\alpha}. \tag{32}$$

[16]For the derivation of them, see Appendix B.

[17]As said in the introduction, what is called the price level both in economics and in general is not (29) but the consumer price index (CPI) which is composed of the price of domestic consumption goods and that of foreign consumption goods. For the CPI in the Keynes model, see Appendix C.

2.4 The Short-Run Equilibrium State

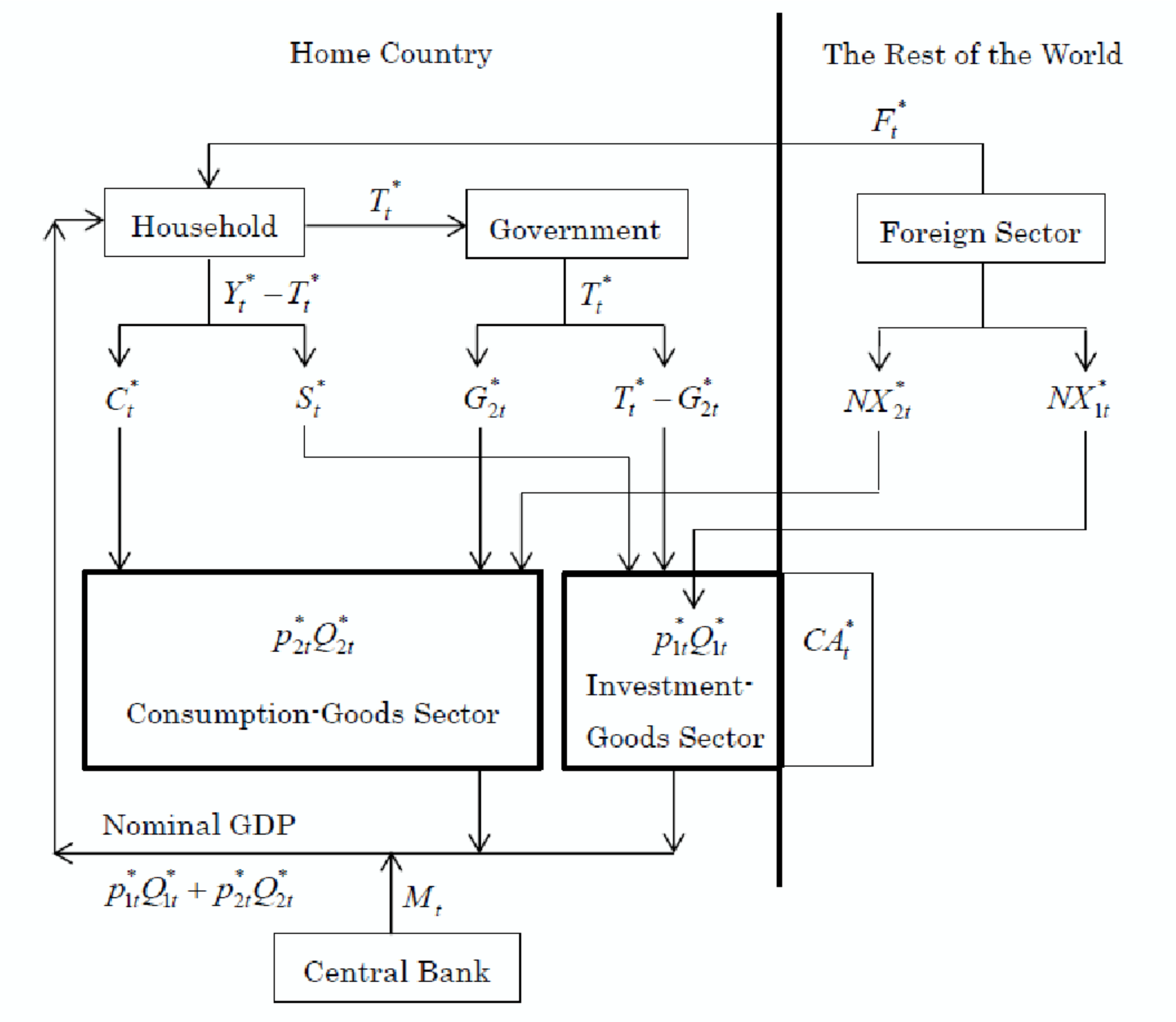

Figure 5. The Short-Run Equilibrium State

As said above, the price p_{1t}^e of investment goods newly produced in the first sub-period of period t is expected at the end of the third sub-period of period $t-1$. How? Although it is not discussed in detail in this paper, at least it is worth mentioning quite a possibility that the price $\tilde{p}_{1t-1}$ of capital stock realized through the portfolio selection during the second sub-period of period $t-1$ may strongly affect the value of p_{1t}^e. In reality p_{1t}^e and $\tilde{p}_{1t-1}$ correspond to the stock prices in the primary and secondary markets of stock, respectively. As is widely recognized, stock prices depend heavily on psychological factors or "animal spirits." Then, they sometimes undergo sharp changes as immediately before and during the Great Depression of the last century and the Great Recession of this century. Thus, it would be a plausible assumption that $\tilde{p}_{1t-1}$ and hence p_{1t}^e sometimes fluctuate suddenly. Anyway, once the value of p_{1t}^e is set, the expected or planned values of other variables such as Q_{1t}^e, p_{2t}^e, and Q_{2t}^e are also calculated on the basis of the Keynes model as in (21), (29), and (30).

But the realization of such a plan is not always warranted. Especially, as suggested in the introduction, money as a medium of exchange matters because more than one kind of goods are traded in this model. Thus, the central bank needs to supply money by the amount calculated as

$$M_t V_t = p^e_{1t} Q^e_{1t} + p^e_{2t} Q^e_{2t}, \tag{33}$$

where V_t is an exogenous variable which is institutionally and socially fixed. If the central bank promises (and keeps its promise) to supply money following (33), the third-sub-period plan is realized in the first sub-period of period t as

$$M_t V_t = p^*_{1t} Q^*_{1t} + p^*_{2t} Q^*_{2t}. \tag{34}$$

Here a superscript $*$ means a realized value. An economy is said to be in the short-run equilibrium state if expected or planned values are all realized. A superscript e attached to variables considered so far is replaced with a superscript $*$ in the short-run equilibrium state as F^*_t, EX^*_{1t}, IM^*_{1t}, EX^*_{2t}, IM^*_{2t}, e^*_t, etc.

(34) appears to suggest the quantity theory of money. But causality runs in the opposite direction in this case. That is, the total amount of aggregate outputs on the right-hand side determines the appropriate quantity of money as a means of payment on the left-hand side. On the other hand, the central bank may decide to supply less money than (33) requires. If so, the initial plan must be revised, i.e., p^e_{1t} must be lowered so that the corresponding planned value on the right-hand side of (33) becomes equal to the left-hand side. In this case, the money supply determines prices as the traditional quantity theory of money says. In either case the supply of and the demand for money coincide in the short-run equilibrium state in the form of (34). A macroeconomy in the short-run equilibrium state can be grasped at once by Figure 5.

3 The Long-Run Model (The Solow Model)

3.1 The Characterization of the Long-Run Equilibrium State

An economy is said to be in the long-run equilibrium state if the following five assumptions are all satisfied:

Assumption 2: The economy is in the short-run equilibrium state.
Assumption 3: Full employment holds in the labor market.
Assumption 4: The rates of return are all equal.
Assumption 5: The long-run price conditions hold.
Assumption 6: The proportionality conditions hold.

Assumption 2 says that *the long-run equilibrium state is a special case of the short-run equilibrium state.* In other words the long-run model is included in the Keynes model as a special case.

Assumption 3 means that

$$N^*_{1t} + N^*_{2t} = N_t, \tag{35}$$

where N_t is the natural level of employment which is assumed to grow at a constant rate n as[18]

$$N_t = (1+n)N_{t-1}. \tag{36}$$

Assumption 4 means the following equality between all the rates of return:

$$h_{1t}^* = i_t = h_{2t}^* = i_{ft}.$$

Assumption 5 means the following relations among various prices:

$$\frac{1}{1-\pi}\, p_{1t-1}^{**} = \frac{1}{1-\pi}\, \tilde{p}_{1t-1}^{**} = p_{1t}^{**} = e_t^{**} p_{f1t}^{**} = \frac{1}{1-\pi} \tilde{p}_{Bt-1}^{**} = \tilde{p}_{Bt}^{**}.$$

A superscript $**$ indicates a value in the long-run equilibrium state. Assumption 5 says that the price p_{1t-1}^{**} of investment goods as flow and the price $\tilde{p}_{1t-1}^{**}$ of investment goods as stock coincide in the same period, and that the expected and realized inflation rate is a constant π which is less than unity by definition.[19] Equality $p_{1t}^{**} = e_t^{**} p_{f1t}^{**}$ implies that the theory of purchasing power parity obtains in the long-run equilibrium state. The price of government bonds is so adjusted by the government as to satisfy Assumption 5. As a result, the bond price rises at the same rate as the price of investment goods.

Assumption 6 means the following proportional relations to capital stock:

$$\begin{aligned}
\frac{G_{1t}^{**}}{p_{1t}^{**}} &= \beta_{G_1} K_t^{**}, \\
\frac{G_{2t}^{**}}{p_{1t}^{**}} &= \beta_{G_2} K_t^{**}, \\
\frac{T_t^{**}}{p_{1t}^{**}} &= \beta_T K_t^{**}, \\
\frac{F_t^{**}}{p_{1t}^{**}} &= \beta_F K_t^{**}, \\
\frac{NX_{1t}^{**}}{p_{1t}^{**}} &= \beta_{NX_1} K_t^{**}, \\
\frac{NX_{2t}^{**}}{p_{1t}^{**}} &= \beta_{NX_2} K_t^{**},
\end{aligned}$$

where K_t^{**} is capital stock in the long-run equilibrium state and coefficients $\beta's$ of K_t^{**} are constants. It can be said economically that $\beta_{G_1} \geq 0$ and $\beta_{G_2} \geq 0$. The signs of other $\beta's$ may be negative. As will be seen below, it is convenient to define β_{BD}, β_{NX}, and β_{CA} as follows:

$$\begin{aligned}
\frac{G_{1t}^{**}}{p_{1t}^{**}} + \frac{G_{2t}^{**}}{p_{1t}^{**}} - \frac{T_t^{**}}{p_{1t}^{**}} &= \beta_{BD} K_t^{**}, \beta_{BD} = \beta_{G_1} + \beta_{G_2} - \beta_T, \\
\frac{NX_{1t}^{**}}{p_{1t}^{**}} + \frac{NX_{2t}^{**}}{p_{1t}^{**}} &= \beta_{NX} K_t^{**}, \beta_{NX} = \beta_{NX_1} + \beta_{NX_2}, \\
\frac{NX_{1t}^{**}}{p_{1t}^{**}} + \frac{NX_{2t}^{**}}{p_{1t}^{**}} + \frac{F_t^{**}}{p_{1t}^{**}} &= \beta_{CA} K_t^{**}, \beta_{CA} = \beta_{NX_1} + \beta_{NX_2} + \beta_F.
\end{aligned}$$

[18] n must be greater than -1. Its admissible value is specified in Section 3.2.

[19] Remember that the expected inflation rate was defined as $\pi_t^e = 1 - (\tilde{p}_{1t-1}/p_{1t}^e)$ in Section 2.3. See also footnote 14.

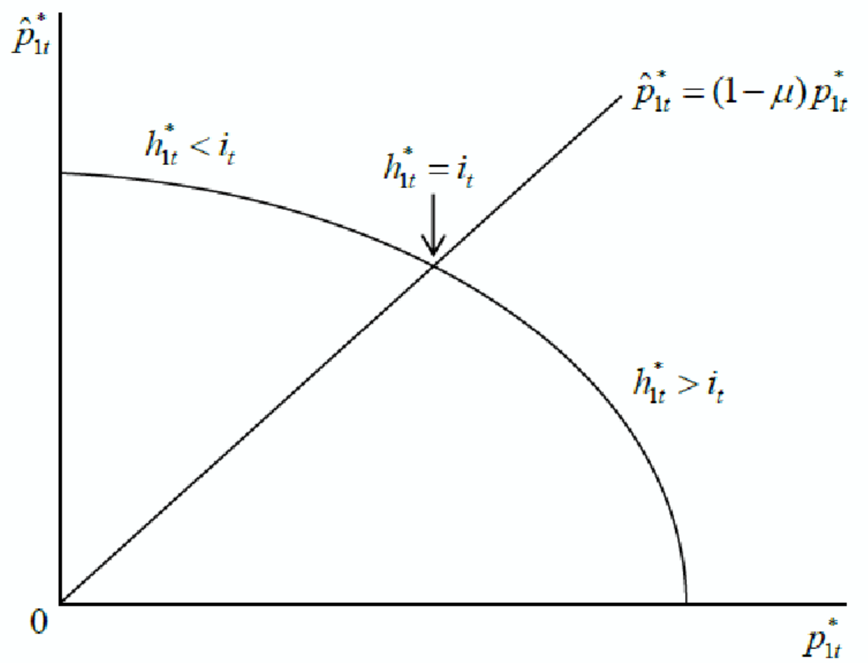

Figure 6. The Normal Supply Price of Investment Goods

Now let us characterize the economy in the long-run equilibrium state using the above assumptions. First, derive the difference between h^e_{1t} and i_t. Rewriting (18) gives

$$(1-\mu)p^e_{1t}Q^e_{1t} = w_t N^e_{1t} + p^e_{1t}(r^e_t + \delta)K_{1t} + (h^e_{1t} - i_t)\tilde{p}_{1t-1}K^h_{1t}, \tag{37}$$

where r^e_t is the real rate of interest defined as

$$r^e_t = \frac{(1+i_t)\tilde{p}_{1t-1}}{p^e_{1t}} - 1. \tag{38}$$

(38) is the Fisher equation. Substituting (20) and (21) into (37) and replacing a superscript e with $*$ yields the difference between h^*_{1t} and i_t:

$$h^*_{1t} - i_t = \frac{p^*_{1t}(r^*_t + \delta)K_{1t}}{\tilde{p}^*_{1t-1}K^h_{1t}} \left\{ \left[\frac{(1-\mu)p^*_{1t}}{\hat{p}^*_{1t}} \right]^{\frac{1}{\alpha}} - 1 \right\}, \tag{39}$$

where

$$\begin{aligned} \hat{p}^*_{1t} &= \left[\frac{(1+i_t)\tilde{p}^*_{1t-1} - (1-\delta)p^*_{1t}}{\alpha} \right]^{\alpha} \left[\frac{w_t}{(1-\alpha)A_t} \right]^{1-\alpha} \\ &= p^*_{1t} \left[\frac{r^*_t + \delta}{\alpha} \right]^{\alpha} \left[\frac{\frac{w_t}{p^*_{1t}}}{(1-\alpha)A_t} \right]^{1-\alpha}. \end{aligned} \tag{40}$$

A price $\hat{p}^*_{1t}$ seems strange. But I like to call it the "normal supply price" of investment goods since Keynes mentioned it in his explanation of production of investment goods.[20] Figure 6 shows the relationship among $\hat{p}^*_{1t}$, p^*_{1t}, h^*_{1t}, and i_t.

[20] In fact Keynes (1936, p. 228) said, "Now those assets [i.e., investment goods] of which the normal supply-price [$\hat{p}^e_{1t}$] is less than the demand-price [$(1-\mu)p^e_{1t}$] will be newly produced; and these will be those assets of which the marginal efficiency [h^e_{1t}] would be greater ... than the rate of interest [i_t]." (Notes in brackets are due to my interpretation. $\hat{p}^e_{1t}$ indicates an expected value of $\hat{p}^*_{1t}$ with p^*_{1t} replaced by p^e_{1t} in (40).)

The difference between h^*_{2t} and i_t can be calculated using (25) and (29) as

$$h^*_{2t} - i_t = \frac{p^*_{1t}(r^*_t + \delta)K_{2t}}{\tilde{p}^*_{1t-1}K^h_{2t}} \left\{ \left[\frac{(1-\mu)p^*_{2t}}{\hat{p}^*_{1t}} \right]^{\frac{1}{\alpha}} - 1 \right\}$$

$$= \frac{p^*_{1t}(r^*_t + \delta)}{\tilde{p}^*_{1t-1}K^h_{2t}} \left\{ \frac{c}{1-c} \left[\frac{(1-\mu)p^*_{1t}}{\hat{p}^*_{1t}} \right]^{\frac{1}{\alpha}} K_{1t} - K_{2t} + \left(\frac{c}{1-c} \frac{F^*_t}{p^*_{1t}} + \frac{1}{1-c} \frac{NX^*_{2t}}{p^*_{1t}} \right) \frac{(1-\mu)\alpha}{r^*_t + \delta} \right\}. \tag{41}$$

Taking account of (39) and the first line of (41), Assumption 4 and Assumption 5 (i.e., $\frac{1}{1-\pi}\, p^{**}_{1t-1} = p^{**}_{1t}$) imply that

$$\frac{1}{1-\pi}\, p^{**}_{1t-1} = p^{**}_{1t} = \frac{1}{1-\mu}\, \hat{p}^{**}_{1t} = p^{**}_{2t}. \tag{42}$$

It is found from (42) that the prices of investment goods and consumption goods coincide and change at the same rate in the long-run equilibrium state. Thus, it is convenient to use only p^{**}_{1t} for them. Then, a nominal value divided by p^{**}_{1t} can be interpreted as a real value in a usual sense. For instance, real GDP Q^{**}_t is expressed as

$$Q^{**}_t = Q^{**}_{1t} + Q^{**}_{2t}, \tag{43}$$

due to (1). Then, using (43), (34) in the short-run equilibrium state becomes the Fisher equation of exchange:

$$M_t V = p^{**}_{1t} Q^{**}_t, \tag{44}$$

where V is the income velocity of money. (44) is of the form of the conventional quantity theory of money. But it should be remembered that (44) is just a special case of (34). As explained in Section 2.4, the money supply may or may not determine prices. Therefore, the quantity theory does not necessarily holds even in the long-run equilibrium state.

When $p^*_{1t} = p^*_{2t} = p^{**}_{1t}$, the labor demand (20) in the investment-goods sector becomes

$$N^*_{1t} = \left[(1-\alpha)A^{1-\alpha}_t \frac{(1-\mu)p^{**}_{1t}}{w_t} \right]^{\frac{1}{\alpha}} K^*_{1t}, \tag{45}$$

and the labor demand (27) in the consumption-goods sector becomes

$$N^*_{2t} = \left[(1-\alpha)A^{1-\alpha}_t \frac{(1-\mu)p^{**}_{1t}}{w_t} \right]^{\frac{1}{\alpha}} K^*_{2t}. \tag{46}$$

Since $K^*_{1t} + K^*_{2t} = K^*_t$, substituting (45) and (46) into (35) yields the following equality:

$$\left[(1-\alpha)A^{1-\alpha}_t \frac{(1-\mu)p^{**}_{1t}}{w_t} \right]^{\frac{1}{\alpha}} K^*_t = N_t.$$

It gives the equilibrium real rate of wage as

$$\frac{w^{**}_t}{p^{**}_{1t}} = (1-\mu)(1-\alpha)A_t \left(\frac{K^{**}_t}{A_t N_t} \right)^{\alpha}. \tag{47}$$

Let capital per unit of effective labor in the right-hand side of (47) be designated by k_t^{**}, and capital per unit of effective labor in the investment-goods sector and capital per unit of effective labor in the consumption-goods sector respectively by k_{1t}^{**} and k_{2t}^{**}:

$$k_t^{**} = \frac{K_t^{**}}{A_t N_t}, \; k_{1t}^{**} = \frac{K_{1t}^{**}}{A_t N_{1t}^{**}}, \text{ and } k_{2t}^{**} = \frac{K_{2t}^{**}}{A_t N_{2t}^{**}},$$

where $N_{1t}^{**} + N_{2t}^{**} = N_t$. Then, (47), (45), and (46) lead to the following equality between three kinds of capital stocks per unit of effective labor:

$$\frac{w_t^{**}}{(1-\mu)p_{1t}^{**}} = (1-\alpha)A_t(k_t^{**})^{\alpha} = (1-\alpha)A_t(k_{1t}^{**})^{\alpha} = (1-\alpha)A_t(k_{2t}^{**})^{\alpha}. \tag{48}$$

(48) shows that in the long-run equilibrium state

$$k_t^{**} = k_{1t}^{**} = k_{2t}^{**}, \tag{49}$$

and $MPL_{1t}^{**} = MPL_{2t}^{**}$ from (22) and (31). Also (49) shows that $MPK_{1t}^{**} = MPK_{2t}^{**}$ from (23) and (32).

In the long-run equilibrium state the real rate of interest (38) is simplified as

$$\begin{aligned} r_t^{**} &= \frac{(1+i_t^{**})\tilde{p}_{t-1}^{**}}{p_{1t}^{**}} - 1 \\ &= (1+i_t^{**})(1-\pi) - 1, \end{aligned} \tag{50}$$

because of Assumption 5. Taking (42) and (48) into account, (40) leads to

$$\frac{r_t^{**}+\delta}{1-\mu} = \alpha(k_t^{**})^{\alpha-1} = \alpha(k_{1t}^{**})^{\alpha-1} = \alpha(k_{2t}^{**})^{\alpha-1}. \tag{51}$$

(50) and (51) require that the central bank should set the nominal rate of interest as

$$i_t^{**} = \frac{1}{1-\pi}\left[(1-\mu)\alpha(k_t^{**})^{\alpha-1} - (\delta-\pi)\right]$$

in order for the economy to be in the long-run equilibrium state.[21]

(42) simplifies (41) with $h_{2t}^{*} = i_t$ as

$$\frac{c}{1-c}K_{1t}^{**} - K_{2t}^{**} + \left(\frac{c}{1-c}\frac{F_t^{**}}{p_{1t}^{**}} + \frac{1}{1-c}\frac{NX_{2t}^{**}}{p_{1t}^{**}}\right)\frac{(1-\mu)\alpha}{r_t^{**}+\delta} = 0.$$

Finally, applying (51) and Assumption 6 to the above result yields the following ratios concerning labor and capital in the long-run equilibrium state:

$$\frac{N_{1t}^{**}}{N_t} = \frac{K_{1t}^{**}}{K_t^{**}} = 1 - c - (c\beta_F + \beta_{NX_2})(k_t^{**})^{1-\alpha}, \tag{52}$$

$$\frac{N_{2t}^{**}}{N_t} = \frac{K_{2t}^{**}}{K_t^{**}} = c + (c\beta_F + \beta_{NX_2})(k_t^{**})^{1-\alpha}, \tag{53}$$

where $K_{1t}^{**} + K_{2t}^{**} = K_t^{**}$.

[21] A policy variable i_t^{**} reminds me of Wicksell's price theory mentioned in the introduction. That is, it is natural to think that if the nominal interest rate i_t is less than (greater) than i_t^{**}, the price $\tilde{p}_{t-1}^{*}$ of capital stock K_{1t} tends to rise (fall) because it is profitable to hold capital stock K_{1t}^{h} as (39) shows. This is not what Wicksell argued, though.

3.2 The Long-Run Steady State

The dynamics of the economy in the long-run equilibrium state continues to be governed by the capital accumulation equation (6) as

$$K_{t+1}^{**} = (1-\delta)K_t^{**} + Q_{1t}^{**} - \frac{EX_{1t}^{**}}{p_{1t}^{**}} + \frac{IM_{1t}^{**}}{e_t^{**}p_{f1t}^{**}}.$$

But Assumptions 5 and 6 simplify it as

$$K_{t+1}^{**} = (1-\delta-\beta_{NX_1})K_t^{**} + Q_{1t}^{**}. \tag{54}$$

Dividing both sides of (54) by effective labor $A_{t+1}N_{t+1}$ in period $t+1$ and considering (17), (36), and (52) gives

$$k_{t+1}^{**} = \frac{1}{1+g_N}\left[(1-\delta-\beta_{CA}+s\beta_F)k_t^{**} + s(k_t^{**})^{\alpha}\right], \tag{55}$$

where s is the gross rate of saving defined as $s = 1-c$, and g_N is the natural rate of growth defined as $g_N = (1+g)(1+n)-1(>-1)$.[22] The long-run model (55) is also called the Solow model here, as compared with the Keynes model as the short-run model the symbol of which is (15) in Section 2. Hence the proposition connecting the short run and the long run follows:

Proposition 4: Under Assumptions 2-6 the Keynes model reduces to the Solow model.

Proposition 4 shows that the neoclassical synthesis is possible not only in a closed economy but also in an open economy.

The economy in the long-run equilibrium state is said to be in the long-run steady state if $k_{t+1}^{**} = k_t^{**}$ in the Solow model (55). So *the long-run steady state is a special case of the long-run equilibrium state*. Let a subscript S denote a value in the long-run steady state in what follows. And assume that

$$\begin{aligned} g_N+\delta+\beta_{NX_1} &> 0, \\ (1-s)(g_N+\delta+\beta_{NX_1}+\beta_F)+\beta_{NX_2} &> 0, \\ g_N+\delta+\beta_{CA} &> 0. \end{aligned}$$

Then, capital k_S^{**} per unit of effective labor in the long-run steady state can easily be calculated from (55) as follows:

$$k_S^{**} = \left(\frac{s}{g_N+\delta+\beta_{CA}-s\beta_F}\right)^{\frac{1}{1-\alpha}}. \tag{56}$$

The first and second assumptions above imply that $g_N+\delta+\beta_{CA}-s\beta_F > 0$ which warrants that k_S^{**} in (56) is a unique positive long-run steady state. Moreover, all three assumptions mean that k_S^{**} is an increasing function of the gross saving rate s since

$$\frac{\partial k_S^{**}}{\partial s} = \frac{1}{1-\alpha}\frac{g_N+\delta+\beta_{CA}}{(g_N+\delta+\beta_{CA}-s\beta_F)^2}\left(\frac{s}{g_N+\delta+\beta_{CA}-s\beta_F}\right)^{\frac{\alpha}{1-\alpha}} > 0.$$

[22] Appendix D shows how to derive (55).

In the long-run steady state,

$$k_S^{**} = k_{S1}^{**} = k_{S2}^{**},$$

due to (49). Thus, the economy in the long-run steady state is characterized by k_S^{**} or K_S^{**}.

As for capital stocks,

$$\begin{aligned}
K_{St}^{**} &= \left(\frac{s}{g_N + \delta + \beta_{CA} - s\beta_F}\right)^{\frac{1}{1-\alpha}} A_t N_t, \\
K_{S1t}^{**} &= \frac{s(g_N + \delta + \beta_{NX_1})}{g_N + \delta + \beta_{CA} - s\beta_F} K_{St}^{**}, && (57) \\
K_{S2t}^{**} &= \frac{(1-s)(g_N + \delta + \beta_{NX_1} + \beta_F) + \beta_{NX_2}}{g_N + \delta + \beta_{CA} - s\beta_F} K_{St}^{**}, && (58)
\end{aligned}$$

because of (56), (52), and (53). K_{St}^{**}, K_{S1t}^{**}, and K_{S2t}^{**} are all positive by the above assumptions.

As for outputs,

$$\begin{aligned}
Q_{S1t}^{**} &= A_t N_{S1t}^{**} (k_{S1}^{**})^{\alpha} \\
&= A_t N_{S1t}^{**} (k_S^{**})^{\alpha} \\
&= (g_N + \delta + \beta_{NX_1}) K_{St}^{**}, && (59) \\
Q_{S2t}^{**} &= A_t N_{S2t}^{**} (k_{S2}^{**})^{\alpha} \\
&= A_t N_{S2t}^{**} (k_S^{**})^{\alpha} \\
&= \frac{(1-s)(g_N + \delta + \beta_{NX_1} + \beta_F) + \beta_{NX_2}}{s} K_{St}^{**}, && (60) \\
Q_{St}^{**} &= Q_{S1t}^{**} + Q_{S2t}^{**} \\
&= \frac{g_N + \delta + \beta_{CA} - s\beta_F}{s} K_{St}^{**}, && (61)
\end{aligned}$$

because of (16), (24), (52), (53), and (43). Q_{S1t}^{**}, Q_{S2t}^{**}, and Q_{St}^{**} are all positive by the assumptions too.

Finally the relationship between private disposable income and consumption is worthy to be examined in the long-run steady state. Real GNI in the long-run steady state is written as

$$\begin{aligned}
\frac{Y_{St}^{**}}{p_{1t}^{**}} &= Q_{S1t}^{**} + Q_{S2t}^{**} + \frac{F_t^{**}}{p_{1t}^{**}} \\
&= \frac{g_N + \delta + \beta_{CA}}{s} K_{St}^{**}, && (62)
\end{aligned}$$

due to (61). Since nominal private disposable income Y_{Dt}^e is defined as $Y_{Dt}^e = Y_t^e - T_t^e - p_{1t}^e(\delta - \pi_t^e)K_t$ on the basis of (1), (18), and (25), its real value in the long-run steady state is calculated as

$$\begin{aligned}
\frac{Y_{DSt}^{**}}{p_{1t}^{**}} &= \frac{Y_{St}^{**}}{p_{1t}^{**}} - \frac{T_t^{**}}{p_{1t}^{**}} - (\delta - \pi) K_{St}^{**} \\
&= \frac{g_N + \delta + \beta_{CA} - s(\delta - \pi + \beta_T)}{s} K_{St}^{**}, && (63)
\end{aligned}$$

using (62). On the other hand, real private consumption in the long-run steady state is written as

$$\frac{C_{St}^{**}}{p_{1t}^{**}} = Q_{S2t}^{**} - \frac{NX_{2t}^{**}}{p_{1t}^{**}} - \frac{G_{2t}^{**}}{p_{1t}^{**}}$$

$$= \frac{(1-s)(g_N+\delta+\beta_{CA}+\beta_{G_2})-\beta_{G_2}}{s}K_{St}^{**}, \tag{64}$$

due to (4) and (60). Dividing (64) by (63) leads to the average propensity to consume as

$$\frac{C_{St}^{**}}{Y_{DSt}^{**}} = \frac{(1-s)(g_N+\delta+\beta_{CA}+\beta_{G_2})-\beta_{G_2}}{g_N+\delta+\beta_{CA}-s(\delta-\pi+\beta_T)}. \tag{65}$$

It follows from (65) that in the long-run steady state the average propensity to consume is a decreasing function of the inflation rate π. In other words, the average propensity to consume takes a constant value if the inflation rate remains unchanged.[23] [24]

4 The FTPL on the KS Model

The macro model constructed so far is able to analyze the short run as the Keynes model and the long run as the Solow model. But it is one after all. Thus it is called the Keynes-Solow model (the KS model, hereafter). Using the KS model, this section examines the FTPL in two respects under the assumption that the FTPL holds in the long-run equilibrium state.

First, let us consider the equilibrium condition for the FTPL. In the KS model the government budget constraint (8) always holds in the short-run equilibrium state by Proposition 1. It also holds in the long-run equilibrium state as follows:

$$G_{1t}^{**}+G_{2t}^{**}+(1+i_t^{**})\tilde{p}_{Bt-1}^{**}B_t^{**} = IT_t^{**}+DT_t^{**}+\tilde{p}_{Bt}^{**}B_{t+1}^{**}. \tag{66}$$

Rewriting (66) gives

$$(1+i_t^{**})\tilde{p}_{Bt-1}^{**}B_t^{**} = PS_t^{**}+\tilde{p}_{Bt}^{**}B_{t+1}^{**}, \tag{67}$$

where PS_t^{**} is the primary surplus defined as

$$PS_t^{**} = IT_t^{**}+DT_t^{**}-(G_{1t}^{**}+G_{2t}^{**}).$$

Dividing both sides of (67) by p_{1t}^{**} yields

$$\frac{(1+i_t^{**})\tilde{p}_{Bt-1}^{**}B_t^{**}}{p_{1t}^{**}} = \frac{PS_t^{**}}{p_{1t}^{**}}+\frac{\tilde{p}_{Bt}^{**}B_{t+1}^{**}}{p_{1t}^{**}}. \tag{68}$$

Forwarding (67) one period leads to

$$(1+i_{t+1}^{**})\tilde{p}_{Bt}^{**}B_{t+1}^{**} = PS_{t+1}^{**}+\tilde{p}_{Bt+1}^{**}B_{t+2}^{**}. \tag{69}$$

[23] Since real private saving S_{DSt}^{**} is defined and calculated as

$$\begin{aligned}\frac{S_{DSt}^{**}}{p_{1t}^{**}} &= \frac{Y_{DSt}^{**}-C_{St}^{**}}{p_{1t}^{**}}\\ &= (g_N+\pi+\beta_{CA}+\beta_{G_2}-\beta_T)K_{St}^{**},\end{aligned}$$

the net rate of saving becomes

$$\frac{S_{DSt}^{**}}{Y_{DSt}^{**}} = \frac{s(g_N+\pi+\beta_{CA}+\beta_{G_2}-\beta_T)}{g_N+\delta+\beta_{CA}-s(\delta-\pi+\beta_T)},$$

which is an increasing function of the inflation rate.

[24] For the analyses of the golden-rule state and the modified-golden-rule state, see Appendices E and F.

Using (69) as well as Assumption 5 (i.e., $p_{1t}^{**} = \tilde{p}_{1t}^{**}$), the second term on the right-hand side of (68) is expressed as

$$\begin{aligned}
\frac{\tilde{p}_{Bt}^{**} B_{t+1}^{**}}{p_{1t}^{**}} &= \frac{\frac{PS_{t+1}^{**}}{p_{1t+1}^{**}}}{\frac{(1+i_{t+1}^{**})p_{1t}^{**}}{p_{1t+1}^{**}}} + \frac{\frac{\tilde{p}_{Bt+1}^{**} B_{t+2}^{**}}{p_{1t+1}^{**}}}{\frac{(1+i_{t+1}^{**})p_{1t}^{**}}{p_{1t+1}^{**}}} \\
&= \frac{\frac{PS_{t+1}^{**}}{p_{1t+1}^{**}}}{\frac{(1+i_{t+1}^{**})\tilde{p}_{1t}^{**}}{p_{1t+1}^{**}}} + \frac{\frac{\tilde{p}_{Bt+1}^{**} B_{t+2}^{**}}{p_{1t+1}^{**}}}{\frac{(1+i_{t+1}^{**})\tilde{p}_{1t}^{**}}{p_{1t+1}^{**}}} \\
&= \frac{\frac{PS_{t+1}^{**}}{p_{1t+1}^{**}}}{1 + r_{t+1}^{**}} + \frac{\frac{\tilde{p}_{Bt+1}^{**} B_{t+2}^{**}}{p_{1t+1}^{**}}}{1 + r_{t+1}^{**}},
\end{aligned} \tag{70}$$

where $PS_{t+1}^{**} = IT_{t+1}^{**} + DT_{t+1}^{**} - (G_{1t+1}^{**} + G_{2t+1}^{**})$ as defined above.

Substituting (70) into (68) yields

$$\frac{(1 + i_t^{**})\tilde{p}_{Bt-1}^{**} B_t^{**}}{p_{1t}^{**}} = \frac{PS_t^{**}}{p_{1t}^{**}} + \frac{\frac{PS_{t+1}^{**}}{p_{1t+1}^{**}}}{1 + r_{t+1}^{**}} + \frac{\frac{\tilde{p}_{Bt+1}^{**} B_{t+2}^{**}}{p_{1t+1}^{**}}}{1 + r_{t+1}^{**}}. \tag{71}$$

Similar calculations extend (71) to

$$\frac{(1 + i_t^{**})\tilde{p}_{Bt-1}^{**} B_t^{**}}{p_{1t}^{**}} = \frac{PS_t^{**}}{p_{1t}^{**}} + \frac{\frac{PS_{t+1}^{**}}{p_{1t+1}^{**}}}{1 + r_{t+1}^{**}} + \frac{\frac{PS_{t+2}^{**}}{p_{1t+2}^{**}}}{(1 + r_{t+1}^{**})(1 + r_{t+2}^{**})} + \frac{\frac{\tilde{p}_{Bt+2}^{**} B_{t+3}^{**}}{p_{1t+2}^{**}}}{(1 + r_{t+1}^{**})(1 + r_{t+2}^{**})},$$

and finally to

$$\frac{(1 + i_t^{**})\tilde{p}_{Bt-1}^{**} B_t^{**}}{p_{1t}^{**}} = \frac{PS_t^{**}}{p_{1t}^{**}} + \sum_{j=1}^{\infty} \frac{\frac{PS_{t+j}^{**}}{p_{1t+j}^{**}}}{(1 + r_{t+1}^{**})...(1 + r_{t+j}^{**})} + \lim_{j \to \infty} \frac{\frac{\tilde{p}_{Bt+j}^{**} B_{t+1+j}^{**}}{p_{1t+j}^{**}}}{(1 + r_{t+1}^{**})...(1 + r_{t+j}^{**})}. \tag{72}$$

(72) seems to be the equilibrium condition for the FTPL. But the KS model says that it is just an artificial equation. In fact,

$$\frac{PS_{t+i}^{**}}{p_{1t+i}^{**}} = (1 + r_{t+i}^{**})B_{t+i}^{**} - B_{t+1+i}^{**}, \ i = 0, 1, 2, ... \ ,$$

due to (68). Then, the right-hand side of (72) reduces to $(1 + r_t^{**})B_t^{**}$ which is exactly the value that the left-hand side takes.[25]

The FTPL is based on a one-good model and argues that the price level of such a single good is so determined as to make the left-hand side and the right-hand side of (72) equal. But such adjusting mechanism of the price level cannot be deduced from the KS model. As

[25] As regards (72), there is an alternative expression as follows:

$$B_t^{**} = \sum_{j=0}^{\infty} \frac{\frac{PS_{t+j}^{**}}{p_{1t+j}^{**}}}{(1 + r_t^{**})...(1 + r_{t+j}^{**})} + \lim_{j \to \infty} \frac{B_{t+1+j}^{**}}{(1 + r_t^{**})...(1 + r_{t+j}^{**})}.$$

Although the two expressions are equivalent mathematically, it is (72) that is used in the FTPL as the equilibrium condition.

has already been explained in detail, the prices of consumption goods and investment goods are determined *by supply of and demand for the goods* as (29) and (7) show. Therefore it is concluded that what is called the equilibrium condition for the FTPL is just an identity.

Second (and independent of the first result), let us examine the convergence of the last term on the right-hand side of (72), the present discounted value of government debt in the limit. In the FTPL it must converge to 0. In order to know whether it tends to 0 with time, divide the numerator and denominator by effective labor $A_{t+1+j}N_{t+1+j}$ in period $t+1+j$ under Assumption 5 (i.e., $p^{**}_{1t+j} = \tilde{p}^{**}_{Bt+j}$). Then,

$$\frac{b^{**}_{t+1+j}}{A_t N_t \left(\frac{1+r^{**}_{t+1}}{1+g_N}\right) \ldots \left(\frac{1+r^{**}_{t+j}}{1+g_N}\right)}, \tag{73}$$

where

$$b^{**}_{t+1+j} = \frac{B^{**}_{t+1+j}}{A_{t+1+j}N_{t+1+j}}.$$

b^{**}_{t+1+j} represents government bonds per unit of effective labor in period $t+1+j$.

In the long-run equilibrium state government bonds evolve according to (66). Dividing both sides of (66) by $p^{**}_{1t}(= \tilde{p}^{**}_{Bt})$ and remembering the definition of the government budget constraint (10) leads to

$$B^{**}_{t+1} = (1-\pi)B^{**}_t + \frac{BD^{**}_t}{p^{**}_{1t}},$$

where $BD^{**}_t = G^{**}_{1t} + G^{**}_{2t} - T^{**}_t$ and $T^{**}_t = IT^{**}_t + DT^{**}_t - d^{**}_t B^{**}_t$. Further dividing both sides of the above difference equation by effective labor $A_{t+1}N_{t+1}$ in period $t+1$ yields the equation for the evolution of government bonds as follows:

$$b^{**}_{t+1} = \frac{1}{1+g_N}\left[(1-\pi)b^{**}_t + \beta_{BD}k^{**}_t\right], \tag{74}$$

where b^{**}_{t+1} and b^{**}_t are government bonds per unit of effective labor in periods $t+1$ and t defined as

$$b^{**}_{t+1} = \frac{B^{**}_{t+1}}{A_{t+1}N_{t+1}}, \text{ and } b^{**}_t = \frac{B^{**}_t}{A_t N_t}.$$

Since it is already known that in the KS model k^{**}_t tends to a constant k^{**}_S given by (56) irrespective of the values of b^{**}_{t+1} and b^{**}_t, it is convenient to focus on the long-run steady state in which case (74) is simplified as

$$b^{**}_{t+1} = \frac{1}{1+g_N}\left[(1-\pi)b^{**}_t + \beta_{BD}k^{**}_S\right]. \tag{75}$$

As is easily seen, a solution to (75) depends on the values of parameters g_N, π, and β_{BD}. Here I take up a "normal" situation in which case β_{BD} is positive. Then, the following proposition follows straightforwardly from (75):

Proposition 5: In the Solow model with $\beta_{BD} > 0$ and $g_N + \pi > 0$, government bonds b^{**}_t per unit of effective labor converges to a positive constant b^{**}_S such that

$$b^{**}_S = \frac{\beta_{BD}}{g_N + \pi}k^{**}_S. \tag{76}$$

Proposition 5 is related only to the numerator of (73). That is, b^{**}_{t+1+j} in the numerator approaches b^{**}_S in (76) as j goes to infinity due to Proposition 5.[26] Then, in order for (73) to converge to 0 the denominator of it must tend to ∞. In the KS model it is known that the real rate r^{**}_{t+j} of interest approaches a constant r^{**}_S such that $r^{**}_S = (1-\mu)\alpha(k^{**}_S)^{\alpha-1} - \delta$ due to (51). Thus, the denominator tends to ∞ if $r^{**}_S > g_N$.[27] These considerations lead to the statement that there exist the conditions on which the present discounted value of government debt converges to 0 in the limit, as the FTPL requires. Explicity they are $\beta_{BD} > 0$, $g_N + \pi > 0$, and $r^{**}_S > g_N$. Nevertheless, this second result does not provide the rational for the FTPL, because the first result above denies it.

5 Conclusions

In the FTPL the discussion concentrates on one simple equation:

$$\frac{\text{Nominal government debt}}{\text{Price level}} = \text{Present value of primary surpluses},$$

where the price level is so determined that both the sides coincide. This logic is parallel to that of the quantity theory of money in which the price level is so adjusted that the following Fisher equation of exchange holds:

$$(\text{Quantity of money}) \times (\text{Income velocity of money}) = (\text{Price level}) \times (\text{Real income}).$$

Which one is right? Or does the truth lie between the two? Is government indispensable for the determination of the price level as the FTPL insists? If so, is there any need to include the foreign sector as well? These questions are very challenging because both the theories are neoclassical. Indeed it can be said that different models come to different conclusions, but it can also be said that after all the situation has not changed since Wicksell cast doubt on the quantity theory more than a century ago, can't it?

[26] From (76) the ratio of capital to government debt in the long-run steady state can be obtained at once as

$$\frac{K^{**}_{St}}{B^{**}_{St}} = \frac{g_N + \pi}{\beta_{BD}}.$$

In addition, using (61) and Assumption 6 the ratio of government debt to real GDP can be calculated as

$$\begin{aligned} \frac{B^{**}_{St}}{Q^{**}_{St}} &= \frac{B^{**}_{St}}{K^{**}_{St}} \frac{K^{**}_{St}}{Q^{**}_{St}} \\ &= \frac{\frac{BD^{**}_S}{p^{**}_{1t} Q^{**}_{St}}}{g_N + \pi}, \end{aligned}$$

where $\frac{BD^{**}_S}{p^{**}_{1t} Q^{**}_{St}} = \frac{s\beta_{BD}}{g_N + \delta + \beta_{CA} - s\beta_F}$. $\frac{BD^{**}_S}{p^{**}_{1t} Q^{**}_{St}}$ is the constant ratio of government deficits to *nominal* GDP, and $g_N + \pi$ is interpreted as the constant growth rate of *nominal* GDP (if π is not far from 0). Thus, the above equation implies that the debt-GDP ratio converges to the ratio of the two constants. This corresponds to the conclusion Domar (1944) obtained in the case of no inflation without using the Solow model.

[27] The value of r^{**}_S depends on that of the saving rate s. For example, it is equal to r^{**}_G in Appendix E, while r^{**}_{MG} in Appendix F.

Motivated under these circumstances, I constructed a macro model which includes the government sector as well as the foreign sector. It is based on both the *General Theory* of Keynes who tried to generalize the quantity theory using flexible prices and the Solow model which does not work without the quantity theory. Such a model is called the KS model. It is a unified model because it is able to analyze both the short run and the long run. In the short run prices and outputs are determined by supply and demand on the basis of the principle of effective demand. In the long run the economy is described by the Solow model and there exists a unique stable steady state. The relationship among the quantity of money, the price level, and real income can be written in the form of the Fisher equation of exchange above. But whether the quantity of money influences the price level depends on the magnitude of nominal outputs on the right-hand side compared with the money supply on the left-hand side. In sum the quantity theory of money cannot be discarded as a "false theory" (Wicksell), but it does not always hold even in the long run.

In the FTPL literature it is often assumed that a single kind of output is given to a consumer every period by a fixed amount and the real rate of interest is also fixed. Thus, I regarded the FTPL as a neoclassical theory and analyzed it in the long-run equilibrium state. Then, using the KS model it has been proved that the equilibrium condition for the FTPL as above is just an identity and it is not related to the determination of the price level of *goods*. It has also been shown that in the KS model there are conditions under which the present discounted value of government debt converges to zero as argued in the FTPL. Although the FTPL made a contribution in that it turned our attention to the role of government debt, the price level is not determined along the line of the FTPL from the perspective of the KS model.

Finally I like to add that the KS model is a model of the neoclassical synthesis. The neoclassical synthesis Samuelson proposed more than 60 years ago was harshly criticized and is almost forgotten now. It went back to the *General Theory*. In fact, Keynes (1936, p. 378) wrote, "But if our central controls succeed in establishing an aggregate volume of output corresponding to full employment as nearly as is practicable, the classical theory comes into its own again from this point onwards." And in the last year of his life he reportedly said, "The long-run mechanisms of the classical system must be allowed to work; but it would only be allowed to work in the long run if short-run aids were supplied." (Harrod (1951, p. 622)) Solow (1970, p. 92) also supported it saying, "There is an additional obvious need for someone to synthesize the theory of growth, which takes full employment for granted, with the shorter-run macroeconomics whose main subject is variation of the volume of employment." Solow (2012, p. 273) repeatedly argues, "How does one make the analytical connection between the short run and the long one? . . . I don't think that that problem is solved and I hope one can continue to try to solve it." I quite agree on the perspective of the neoclassical synthesis and I think that it is possible. The KS model is intended for it and it provides a long-run macro model which has the short-run foundations.

Appendix

A Supply Curve of Domestic Investment Goods

As said in Section 2.2, (21) is a supply curve of domestic investment goods. To express it in a usual way, replace Q_{1t}^e and p_{1t}^e in (21) respectively with Q_{1t}^S and p_{1t}. Then,

$$Q_{1t}^S = p_{1t}^{\frac{1-\alpha}{\alpha}} \left[\frac{(1-\mu)(1-\alpha)A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}} K_{1t}.$$

To examine the shape of the graph on the Q_{1t}-p_{1t} plane, differentiate Q_{1t}^S w.r.t. p_{1t} once and twice. Then,

$$\frac{dQ_{1t}^S}{dp_{1t}} = \frac{1-\alpha}{\alpha} p_{1t}^{\frac{1-2\alpha}{\alpha}} \left[\frac{(1-\mu)(1-\alpha)A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}} K_{1t} > 0,$$

and

$$\frac{d^2Q_{1t}^S}{dp_{1t}^2} = \frac{1-\alpha}{\alpha}\frac{1-2\alpha}{\alpha} p_{1t}^{\frac{1-3\alpha}{\alpha}} \left[\frac{(1-\mu)(1-\alpha)A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}} K_{1t} \begin{cases} > 0 & \text{if } 0 < \alpha < \frac{1}{2} \\ = 0 & \text{if } \alpha = \frac{1}{2} \\ < 0 & \text{if } \frac{1}{2} < \alpha < 1. \end{cases}$$

The shape of a supply curve in Figure 3 is based on a usual assumption that α is around one third.

B Supply and Demand Curves of Domestic Consumption Goods

As said in Section 2.2, (28) is a supply curve of domestic consumption goods. To express it in a usual way, replace Q_{2t}^e and p_{2t}^e in (28) respectively with Q_{2t}^S and p_{2t}. Then,

$$Q_{2t}^S = p_{2t}^{\frac{1-\alpha}{\alpha}} \left[\frac{(1-\mu)(1-\alpha)A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}} K_{2t}.$$

As is obvious, the argument on Q_{1t}^S in Appendix A applies to that on Q_{2t}^S in the same fashion.

Then, let us move on to the consumption-goods demand curve. In order to derive it as a function of p_{2t}, first notice that the right-hand side of (4) is the total expenditure on domestic consumption goods. Second, using (13) and (1), write it as a function of Q_{2t}^S as follows:

$$\begin{aligned} C_t^e + G_{2t}^e + NX_{2t}^e &= cY_t^e + NX_{2t}^e \\ &= c(p_{1t}^e Q_{1t}^e + p_{2t}^e Q_{2t}^e + F_t^e) + NX_{2t}^e \\ &= c(p_{1t}^e Q_{1t}^e + p_{2t} Q_{2t}^S + F_t^e) + NX_{2t}^e. \end{aligned}$$

Lastly the demand Q_{2t}^D for domestic consumption goods is obtained by dividing the above expenditure by the price p_{2t}:

$$Q_{2t}^D = cQ_{2t}^S + \frac{c(p_{1t}^e Q_{1t}^e + F_t^e) + NX_{2t}^e}{p_{2t}},$$

where $p_{1t}^e Q_{1t}^e$, F_t^e, and NX_{2t}^e are given, and Q_{2t}^S is a function of p_{2t} as above.

In order to know the shape of the demand curve on the Q_{2t}-p_{2t} plane, differentiate Q_{2t}^D w.r.t. p_{2t} once and twice. Then,

$$\frac{dQ_{2t}^D}{dp_{2t}} = c\frac{1-\alpha}{\alpha} p_{2t}^{\frac{1-2\alpha}{\alpha}} \left[\frac{(1-\mu)(1-\alpha)A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}} K_{2t} - \frac{c(p_{1t}^e Q_{1t}^e + F_t^e) + NX_{2t}^e}{p_{2t}^2},$$

and

$$\frac{d^2Q_{2t}^D}{dp_{2t}^2} = c\frac{1-\alpha}{\alpha}\frac{1-2\alpha}{\alpha}p_{2t}^{\frac{1-3\alpha}{\alpha}}\left[\frac{(1-\mu)(1-\alpha)A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}}K_{2t} + \frac{2[c(p_{1t}^eQ_{1t}^e + F_t^e) + NX_{2t}^e]}{p_{2t}^3} > 0.$$

It follows from these results that demand curve Q_{2t}^D is bending forward and that it changes the sign of the slope at $p_{2t} = \bar{p}_{2t}$, where

$$\bar{p}_{2t} = \left[\frac{\alpha}{1-\alpha}\frac{1-c}{c}\right]^{\alpha}\left[\frac{w_t}{(1-\mu)(1-\alpha)A_t}\right]^{1-\alpha}\left[\frac{1}{K_{2t}}\right]^{\alpha}\left[\frac{c(p_{1t}^eQ_{1t}^e + F_t^e) + NX_{2t}^e}{1-c}\right]^{\alpha}.$$

If $\alpha < c$ as assumed in usual macroeconomics, $\bar{p}_{2t}$ is smaller than p_{2t}^e in (29) as Figure 4 shows.

C CPI in the Keynes Model

In the Keynes model the (expected) CPI p_t^e can be defined as

$$p_t^e = \frac{C_t^e + G_{2t}^e}{(C_t^e + G_{2t}^e)|_{\text{real}}},$$

where the numerator represents nominal national consumption and the denominator represents real national consumption. The former is calculated from (4) as

$$C_t^e + G_{2t}^e = p_{2t}^eQ_{2t}^e - EX_{2t}^e + IM_{2t}^e,$$

while the latter is defined as

$$(C_t^e + G_{2t}^e)|_{\text{real}} = Q_{2t}^e - \frac{EX_{2t}^e}{p_{2t}^e} + \frac{IM_{2t}^e}{e_t^ep_{f2t}^e},$$

where p_{f2t}^e is the (expected) price of foreign consumption goods in foreign currency.

Substituting these into the above definition of the CPI and rearranging gives

$$p_t^e = (1-\theta_t)p_{2t}^e + \theta_te_t^ep_{f2t}^e,$$

where $\theta_t = \frac{\frac{IM_{2t}^e}{e_t^ep_{f2t}^e}}{(C_t^e+G_{2t}^e)|_{\text{real}}}$. θ_t is the share of imported foreign consumption goods among real national consumption. Thus, the CPI p_t^e is a weighted average of the price p_{2t}^e of domestic consumption goods and the price $e_t^ep_{f2t}^e$ of foreign consumption goods in domestic currency. Do not forget that p_{2t}^e is determined endogenously as in (29). It is assumed in this paper that in the long run $p_{2t}^e = e_t^ep_{f2t}^e$, then $p_t^e = p_{2t}^e$.

D Derivation of Capital Accumulation Equation (55)

Dividing both sides of (54) by $A_{t+1}N_{t+1}$ and using (52) leads to

$$k_{t+1}^{**} = \frac{1-\ \delta-\beta_{NX_1}}{(1+g)(1+n)}k_t^{**} + \frac{A_tN_{1t}^{**}}{A_{t+1}N_{t+1}}(k_{1t}^{**})^{\alpha}$$

$$
\begin{aligned}
&= \frac{1-\delta-\beta_{NX_1}}{(1+g)(1+n)}k_t^{**} + \frac{1}{(1+g)(1+n)}\frac{N_{1t}^{**}}{N_t}(k_{1t}^{**})^{\alpha} \\
&= \frac{1-\delta-\beta_{NX_1}}{1+g_N}k_t^{**} + \frac{1}{1+g_N}[1-c-(c\beta_F+\beta_{NX_2})(k_t^{**})^{1-\alpha}](k_t^{**})^{\alpha} \\
&= \frac{1-\delta-\beta_{NX_1}-(1-s)\beta_F-\beta_{NX_2}}{1+g_N}k_t^{**} + \frac{1}{1+g_N}s(k_t^{**})^{\alpha} \\
&= \frac{1}{1+g_N}\left[(1-\delta-\beta_{NX}-\beta_F+s\beta_F)k_t^{**}+s(k_t^{**})^{\alpha}\right] \\
&= \frac{1}{1+g_N}\left[(1-\delta-\beta_{CA}+s\beta_F)k_t^{**}+s(k_t^{**})^{\alpha}\right],
\end{aligned}
$$

where $s = 1-c$ and $g_N = (1+g)(1+n)-1$.

E Analysis of the Golden-Rule State

The golden-rule state is defined as the long-run steady state in which real national consumption $\frac{C_{St}^{**}+G_{2St}^{**}}{p_{1t}^{**}}$ is maximized every period. Remembering the consumption function (13) and using (62) gives real national consumption as a function of the gross rate s of saving:

$$
\begin{aligned}
\frac{C_{St}^{**}+G_{2St}^{**}}{p_{1t}^{**}} &= \frac{(1-s)Y_{St}^{**}}{p_{1t}^{**}} \\
&= (1-s)\left(\frac{g_N+\delta+\beta_{CA}}{s}\right)K_{St}^{**} \\
&= (1-s)\left(\frac{g_N+\delta+\beta_{CA}}{s}\right)\left(\frac{s}{g_N+\delta+\beta_{CA}-s\beta_F}\right)^{\frac{1}{1-\alpha}}A_tN_t. \qquad (77)
\end{aligned}
$$

Let s_G be the saving rate which maximizes the real national consumption. It can be obtained from the differentiation of (77) with respect to s.[28] Hence,

$$
\begin{aligned}
s_G &= \frac{\alpha(g_N+\delta+\beta_{CA})}{g_N+\delta+\beta_{NX}+\alpha\beta_F} \\
&= \alpha\left[1+\frac{(1-\alpha)\beta_F}{g_N+\delta+\beta_{NX}+\alpha\beta_F}\right]. \qquad (78)
\end{aligned}
$$

The corresponding ratio c_G of national consumption to GNI is as follows:

$$
\begin{aligned}
c_G &= \frac{(1-\alpha)(g_N+\delta+\beta_{NX})}{g_N+\delta+\beta_{NX}+\alpha\beta_F} \\
&= (1-\alpha)\left[1-\frac{\alpha\beta_F}{g_N+\delta+\beta_{NX}+\alpha\beta_F}\right].
\end{aligned}
$$

Let a subscript G in place of S denote a value in the golden-rule state. Then, k_S^{**} in (56) is written as

$$
k_G^{**} = \left(\frac{\alpha}{g_N+\delta+\beta_{NX}}\right)^{\frac{1}{1-\alpha}}. \qquad (79)
$$

[28] $\frac{C_{St}^{**}+G_{2St}^{**}}{p_{1t}^{**}}$ is equal to $Q_{S2t}^{**}-\beta_{NX_2}K_{St}^{**}$ because of (4). Then, s_G can also be calculated using the following relation:

$$
Q_{S2t}^{**}-\beta_{NX_2}K_{St}^{**} = [(k_S^{**})^{\alpha}-(g_N+\delta+\beta_{NX})\;k_S^{**}\;]A_tN_t,
$$

which is derived from (59) - (61).

by substituting (78) into (56).

As for capital stocks,

$$K_{Gt}^{**} = \left(\frac{\alpha}{g_N + \delta + \beta_{NX}}\right)^{\frac{1}{1-\alpha}} A_t N_t, \tag{80}$$

$$K_{G1t}^{**} = \left(\alpha - \frac{\alpha\ \beta_{NX_2}}{g_N + \delta + \beta_{NX}}\right) K_{Gt}^{**}, \tag{81}$$

$$K_{G2t}^{**} = \left[(1-\alpha) + \frac{\alpha\ \beta_{NX_2}}{g_N + \delta + \beta_{NX}}\right] K_{Gt}^{**}, \tag{82}$$

because of (79), (57), and (58) along with (78).

As for outputs,

$$Q_{G1t}^{**} = (g_N + \delta + \beta_{NX_1}) K_{Gt}^{**}, \tag{83}$$

$$Q_{G2t}^{**} = \frac{(1-\alpha)(g_N + \delta + \beta_{NX}) + \alpha\ \beta_{NX_2}}{\alpha} K_{Gt}^{**}, \tag{84}$$

$$Q_{Gt}^{**} = \frac{g_N + \delta + \beta_{NX}}{\alpha} K_{Gt}^{**}, \tag{85}$$

because of (59) - (61) and (80) along with (78).[29]

Real GNI in the golden-rule state is written as

$$\frac{Y_{Gt}^{**}}{p_{1t}^{**}} = \frac{g_N + \delta + \beta_{NX} + \alpha\beta_F}{\alpha} K_{Gt}^{**},$$

because of (85). Then, real private disposable income is calculated by definition as

$$\frac{Y_{DGt}^{**}}{p_{1t}^{**}} = \frac{g_N + \delta + \beta_{NX} - \alpha(\delta - \pi + \beta_T - \beta_F)}{\alpha} K_{Gt}^{**}.$$

Because real private consumption is written as

$$\frac{C_{Gt}^{**}}{p_{1t}^{**}} = \frac{(1-\alpha)(g_N + \delta + \beta_{NX}) - \alpha\beta_{G_2}}{\alpha} K_{Gt}^{**},$$

the average propensity to consume in the golden-rule becomes

$$\begin{aligned} \frac{C_{Gt}^{**}}{Y_{DGt}^{**}} &= \frac{(1-\alpha)(g_N + \delta + \beta_{NX}) - \alpha\beta_{G_2}}{g_N + \delta + \beta_{NX} - \alpha(\delta - \pi + \beta_T - \beta_F)} \\ &= (1-\alpha) + \frac{\alpha(1-\alpha)(\delta - \pi + \beta_T - \beta_F - \beta_{G_2})}{g_N + \delta + \beta_{NX} - \alpha(\delta - \pi + \beta_T - \beta_F)} \gtreqless 1 - \alpha, \end{aligned}$$

[29] As to the ratio of the consumption-goods sector to the investment-goods sector,

$$\frac{K_{G2t}^{**}}{K_{G1t}^{**}} = \frac{Q_{G2t}^{**}}{Q_{G1t}^{**}} = \frac{1-\alpha}{\alpha} + \frac{\beta_{NX_2}}{\alpha(g_N + \delta + \beta_{NX_1})},$$

because of (81) - (84). And in terms of the capital-output ratio, (85) becomes

$$\frac{K_{Gt}^{**}}{Q_{Gt}^{**}} = \frac{\alpha}{g_N + \delta + \beta_{NX}}.$$

which is a decreasing function of the inflation rate.

Similarly, since real private saving is written as

$$\frac{S_{DGt}^{**}}{p_{1t}^{**}} = (g_N + \pi + \beta_{CA} + \beta_{G_2} - \beta_T)K_{Gt}^{**},$$

the net rate of saving is calculated as

$$\frac{S_{DGt}^{**}}{Y_{DGt}^{**}} = \frac{\alpha(g_N + \pi + \beta_{CA} + \beta_{G_2} - \beta_T)}{g_N + \delta + \beta_{NX} - \alpha(\delta - \pi + \beta_T - \beta_F)} \lesseqgtr \alpha.$$

Finally the ratio of national consumption to GDP becomes

$$\frac{C_{Gt}^{**} + G_{2t}^{**}}{p_{1t}^{**}Q_{Gt}^{**}} = 1 - \alpha,$$

while the real rate of interest is calculated as

$$r_G^{**} = (1 - \mu)(g_N + \delta + \beta_{NX}) - \delta.$$

F Analysis of the Modified-Golden-Rule State

The modified-golden-rule state here is defined as the long-run steady state in which the sum of present discounted values of utility of the household sector is maximized as follows:

$$\max_s \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t \frac{(c_{St}^{**})^{1-\gamma}}{1-\gamma} N_t, \ \gamma > 0,$$

$$\text{s.t.} \sum_{t=0}^{\infty} \frac{c_{St}^{**} N_t}{(1+r_{S0}^{**})...(1+r_{St}^{**})} + \lim_{t\to\infty} \frac{K_{St}^{**} + B_{fSt}^{**}}{(1+r_0^{**})...(1+r_t^{**})} = K_{S0}^{**} + B_{fS0}^{**} + \sum_{t=0}^{\infty} \frac{\frac{w_{St}^{**}}{p_{1t}^{**}} N_t + \mu Q_{St}^{**}}{(1+r_0^{**})...(1+r_t^{**})},$$

where

$$\begin{aligned} c_{St}^{**} &= \frac{\frac{C_{St}^{**} + G_{2t}^{**}}{p_{1t}^{**}}}{N_t} \\ &= (1-s)\left(\frac{g_N + \delta + \beta_{CA}}{s}\right)\left(\frac{s}{g_N + \delta + \beta_{CA} - s\beta_F}\right)^{\frac{1}{1-\alpha}} A_t, \end{aligned}$$

due to (77). The intertemporal budget constraint is obtained by the iteration of the budget constraint in period t:

$$K_{t+1}^{**} + B_{ft+1}^{**} = (1 + r_t^{**})(K_t^{**} + B_{ft}^{**}) + \left[\frac{w_t^{**}}{p_{1t}^{**}} N_t + \mu Q_t^{**} - \left(\frac{C_t^{**}}{p_{1t}^{**}} + \frac{G_{2t}^{**}}{p_{1t}^{**}}\right)\right].$$

A difference from an ordinary setting is that the utility maximization problem above is confined to the long-run steady state. This idea is based on two reasons. The first is that the analysis becomes much easier. As far as the saving rate s is fixed, the existence and uniqueness of a stable steady state of the KS model is guaranteed as in (56). No transversality condition is required, or no complicated explanation of solution paths on the phase plane is needed. The second is that a dynamically inefficient economy can emerge in addition to a dynamically efficient economy. As is well known, the former can be examined only by the overlapping-generations model. The Ramsey-Cass-Koopmans model with an infinite horizon can deal with the latter only. The KS model with an infinite horizon is able to analyze both the situations.

The Euler equation related to the above utility maximization problem is written as

$$\frac{c^{**}_{St+1}}{c^{**}_{St}} = \left(\frac{1 + r^{**}_{St+1}}{1+\rho}\right)^{\frac{1}{\gamma}},$$

where

$$r^{**}_{St+1} = (1-\mu)\frac{\alpha(g_N + \delta + \beta_{CA} - s\beta_F)}{s} - \delta,$$

because of (51) and (56). Solving it gives the utility maximizing rate s_{MG} of saving as

$$s_{MG} = \frac{(1-\mu)\alpha(g_N + \delta + \beta_{CA})}{(1+\rho)(1+g)^{\gamma} - 1 + \delta + (1-\mu)\alpha\beta_F}, \tag{86}$$

and the corresponding ratio c_{MG} of national consumption to GNI as

$$c_{MG} = \frac{(1+\rho)(1+g)^{\gamma} - 1 + \delta - (1-\mu)\alpha(g_N + \delta + \beta_{NX})}{(1+\rho)(1+g)^{\gamma} - 1 + \delta + (1-\mu)\alpha\beta_F}.$$

$s_G \gtreqless s_{MG}$ for $(1+\rho)(1+g)^{\gamma} - 1 + \delta \gtreqless (1-\mu)(g_N + \delta + \beta_{NX})$, because

$$s_G - s_{MG} = \frac{\alpha(g_N + \delta + \beta_{CA})[(1+\rho)(1+g)^{\gamma} - 1 + \delta - (1-\mu)(g_N + \delta + \beta_{NX})]}{(g_N + \delta + \beta_{NX} + \alpha\beta_F)[(1+\rho)(1+g)^{\gamma} - 1 + \delta + (1-\mu)\alpha\beta_F]}.$$

Let a subscript MG in place of S denote a value in the modified-golden-rule state. Then, k^{**}_S in (56) is written as

$$k^{**}_{MG} = \left[\frac{(1-\mu)\alpha}{(1+\rho)(1+g)^{\gamma} - 1 + \delta}\right]^{\frac{1}{1-\alpha}}. \tag{87}$$

by substituting (86) into (56). $k^{**}_G \gtreqless k^{**}_{MG}$ for $(1+\rho)(1+g)^{\gamma} - 1 + \delta \gtreqless (1-\mu)(g_N + \delta + \beta_{NX})$, because

$$k_G^{1-\alpha} - k_{MG}^{1-\alpha} = \frac{\alpha[(1+\rho)(1+g)^{\gamma} - 1 + \delta - (1-\mu)(g_N + \delta + \beta_{NX})]}{(g_N + \delta + \beta_{NX})[(1+\rho)(1+g)^{\gamma} - 1 + \delta]}.$$

As for capital stocks,

$$\begin{aligned}
K^{**}_{MGt} &= \left[\frac{(1-\mu)\alpha}{(1+\rho)(1+g)^{\gamma} - 1 + \delta}\right]^{\frac{1}{1-\alpha}} A_t N_t, \\
K^{**}_{MG1t} &= \frac{(1-\mu)\alpha(g_N + \delta + \beta_{NX_1})}{(1+\rho)(1+g)^{\gamma} - 1 + \delta} K^{**}_{MGt}, \\
K^{**}_{MG2t} &= \left[1 - \frac{(1-\mu)\alpha(g_N + \delta + \beta_{NX_1})}{(1+\rho)(1+g)^{\gamma} - 1 + \delta}\right] K^{**}_{MGt},
\end{aligned} \tag{88}$$

because of (87), (57), and (58) along with (86).

As for outputs,

$$
\begin{aligned}
Q^{**}_{MG1t} &= (g_N+\delta+\beta_{NX_1})K^{**}_{MGt},\\
Q^{**}_{MG2t} &= \frac{(1+\rho)(1+g)^{\gamma}-1+\delta-(1-\mu)\alpha(g_N+\delta+\beta_{NX_1})}{(1-\mu)\alpha}K^{**}_{MGt},\\
Q^{**}_{MGt} &= \frac{(1+\rho)(1+g)^{\gamma}-1+\delta}{(1-\mu)\alpha}K^{**}_{MGt}, \qquad (89)
\end{aligned}
$$

because of (59) - (61) and (88) along with (86).[30]

Real GNI in the modified-golden-rule state is written as

$$\frac{Y^{**}_{MGt}}{p^{**}_{1t}} = \frac{(1+\rho)(1+g)^{\gamma}-1+\delta+(1-\mu)\alpha\beta_F}{(1-\mu)\alpha}K^{**}_{MGt},$$

because of (89). Then, real private disposable income is calculated as

$$\frac{Y^{**}_{DMGt}}{p^{**}_{1t}} = \frac{(1+\rho)(1+g)^{\gamma}-1+\delta-(1-\mu)\alpha(\delta-\pi+\beta_T-\beta_F)}{(1-\mu)\alpha}K^{**}_{MGt}.$$

Because real private consumption is written as

$$\frac{C^{**}_{MGt}}{p^{**}_{1t}} = \frac{(1+\rho)(1+g)^{\gamma}-1+\delta-(1-\mu)\alpha(g_N+\delta+\beta_{NX}+\beta_{G_2})}{(1-\mu)\alpha}K^{**}_{MGt},$$

the average propensity to consume in the modified-golden-rule becomes

$$\frac{C^{**}_{MGt}}{Y^{**}_{DGt}} = \frac{(1+\rho)(1+g)^{\gamma}-1+\delta-(1-\mu)\alpha(g_N+\delta+\beta_{NX}+\beta_{G_2})}{(1+\rho)(1+g)^{\gamma}-1+\delta-(1-\mu)\alpha(\delta-\pi+\beta_T-\beta_F)},$$

which is a decreasing function of the inflation rate.

Since real private saving is written as

$$\frac{S^{**}_{DMGt}}{p^{**}_{1t}} = (g_N+\pi+\beta_{CA}+\beta_{G_2}-\beta_T)K^{**}_{MGt},$$

the net rate of saving is calculated as

$$\frac{S^{**}_{DMGt}}{Y^{**}_{DMGt}} = \frac{(1-\mu)\alpha(g_N+\pi+\beta_{CA}+\beta_{G_2}-\beta_T)}{(1+\rho)(1+g)^{\gamma}-1+\delta-(1-\mu)\alpha(\delta-\pi+\beta_T-\beta_F)}.$$

Finally the ratio of national consumption to GDP becomes

$$\frac{C^{**}_{MGt}+G^{**}_{2t}}{p^{**}_{1t}Q^{**}_{MGt}} = 1-\frac{(1-\mu)(g_N+\delta+\beta_{NX})}{(1+\rho)(1+g)^{\gamma}-1+\delta}\alpha,$$

while the real rate of interest is calculated as

$$
\begin{aligned}
r^{**}_{MG} &= (1+\rho)(1+g)^{\gamma}-1\\
&\approx \gamma g+\rho,
\end{aligned}
$$

which is the natural rate of interest (due to Wicksell).

[30] In terms of the capital-output ratio, (89) becomes

$$\frac{K^{**}_{MGt}}{Q^{**}_{MGt}} = \frac{(1-\mu)\alpha}{(1+\rho)(1+g)^{\gamma}-1+\delta}.$$

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