Product Boundary, Vertical Competition, and the Double Mark-up Problem[+]

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Abstract

In the computing industry, computing systems typically consist of multiple components supplied by independent firms. Firms must cooperate with each other in "making a system work," but at the same time compete for dividing the industry profits. A firm may, by incorporating the functions of other firms' components into its own product, make its product less dependent on other firms providing complementary components. We analyze how one firm's product boundary expansion affects pricing, profits, social welfare, and the other firm's R&D incentive.

Keywords: Vertical Competition, Product Boundary, Double Mark-up

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1. Introduction

In the computing industry, there has been no single dominant vertically integrated firm since 1990. Instead, the industry is characterized by vertical disintegration i.e., computer systems or platforms consist of many vertically related layers of components supplied by different firms. Because of complementarity, firms in different layers rely on one another, but at the same time they compete against each other for a bigger share of the industry profits.

To ensure a larger share of industry profits, firms try to make their products less dependent on providers of complementary products. As Bresnahan (1999) points out, a firm producing one product has an incentive to enter the other 'turf' by incorporating functions provided by the other firms. For example, in its early days, MS Windows did not include program functions such as WordPad, Internet Explorer (I.E.), and Windows Media. But over time it has included these and other programs that were previously supplied by independent firms. Another example is secondary cache. Once a separate piece of hardware, secondary cache is now integrated into the Intel CPU.

As firms constantly try to expand their product boundaries, the boundaries between adjacent layers and the relationships among those products change continuously as a consequence of both technological innovation and vertical competition. It is important to understand complementarity and vertical competition among different layers of the computing industry.

In 1838, Cournot analysed the pricing of products that are symmetrically complementary with each other, like left and right shoes. However, the current complementary relationship in the computing industry is quite different from the symmetric complementarity analysed by Cournot and others. In many cases, the complementarity among products is asymmetric. For instance, an advanced application program enhances the value of an operating system (O/S), but it is useless without the O/S. In contrast, the O/S provides its basic functions without the advanced application program.

In this paper we analyse the strategic interactions between two firms when the complementarity between these firms' products is asymmetric. In our model, there are two firms, A and B, that provide complementary products A and B, respectively. Product A by itself provides a quality of z, but consumers derive a higher performance quality of q (i.e., $q > z$) by combining the two products. Product B does not provide any function without product A, but enhances the latter's performance from z to q

We first analyze the pricing game between firms A and B for a given z, 0< z <q. Then, when firm A expands its function by increasing z, we examine the effects. We find that firm A's optimal price might not be continuous in z; firm A's profit increases continuously in z, while firm B's profit exhibits an upward jump at a certain level of z, but afterwards decreases as z increases. If z is low, then the firms' individual profit-maximizing prices yield the standard characteristics of complementary goods. An interesting result in our paper is that there exists a critical value of z, say z*, strictly between 0 and q such that if z is larger than z*, then the two firms' prices result in a region in which their products are independent of each other. That is, each firm's demand depends only on its own price, not on the price of the other product. As a result, the two firms behaves as if they are independent firms.

Interestingly enough, in this range of z, the two firms' independently determined prices maximize their joint profits. That is to say, the well-known "double-markup" problem disappears completely, even though the two products are still complementary to a certain degree (i.e., z*< q). Because the double mark-up problem vanishes, there is a region in which an increase in z has a positive effect on firm B's profit as well as on consumers' surplus, thus yielding a win-win situation for both firms and consumers.

Related Literature

There are several recent studies on complementary technologies and patents (see Farrell and Katz 2000, Gilbert 2002, Lerner and Tirole 2002, Shapiro 2001 among others). Farrell and Katz (2000) analyze the incentive of a monopolist in product A to enter complementary product B's market in order to force independent suppliers of B to charge lower prices, which increases its own profits made from product A. If consumers in our model were homogeneous, then it would become very similar to that of Farrell and Katz (2000): an increase in z "price squeezes" product B and always has negative effects on firm B's profit. With heterogeneous consumer preference, however, we show that an increase in z can have a positive effect on firm B's profit. Also, the double mark-up problem disappears at a critical level of z that is strictly less than q.

In a model of patent portfolios that allows a full range of complementarity and substitutability, Lerner and Tirole (2004) analyze the welfare effect of patent pools and evaluate several factors that encourage or hinder the formation of these pools.[1]

Product A in our model can be regarded as a bundle of two complementary products, A_1 and B_1, i.e., A_1 and B_1 combine to yield a performance quality z whereas A_1 and B combine to yield a performance quality q.[2] Thus, our paper is related to the literature on tying/bundling. (See Whinston (1990), Choi and Stefanadis (2001), Carlton and Waldman (2002), Gilbert and Riordan (2002), and Nalebuff (2004), among others.)

In Whinston (1990), Choi and Stefanadis (2001), and Carlton and Waldman (2002), when an incumbent ties its products, consumes cannot undo the tie and cannot add an entrant's product to the tied one.[3] However, when Microsoft ties its Windows O/S

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[1] Their model is closely related to our model in the sense that the degree of complementarity between patents is endogenously determined by licensing fees. In their model all users or licensees derive the same amount of marginal benefits from an additional patent. However, in our model consumers of different types derive different amounts of marginal benefits from an additional product. Because of the different assumptions, we can get the independency result if z is sufficiently high.

[2] If firm A sets p_A first, and, subsequently, firms A and B set prices for B1 and B simultaneously, then it can be shown that unbundling will have no effect on our results because the equilibrium price of B1 is zero.

[3] Whinston (1990) shows that tying, as a commitment to aggressive pricing, could deter new entry. In Choi and Stefanadis (2001), when an incumbent ties products A and B, an entrant into the market for product A can sell its product only if other firms provide complementary product B. As a result, tying can make the prospect of successful entry less certain. In Carlton and Waldman (2002), product A is essential in period 1. However, if a firm enters market B in period 1, it can develop another variety of A in period 2. To protect its profits made from A, the incumbent ties the two products to deter entry into market B.

and its applications such as I.E., Windows still works with Netscape.[4] We regard the tying of I.E. as an expansion of Microsoft's product boundary, which allows consumers to add a rival product to its OS.

Nalebuff (2004) shows that when consumers are heterogenous in their valuations of products A and B, an incumbent, by bundling A and B, can significantly lower the profits of a single-product entrant. Thus, bundling could be quite an effective entry deterrent strategy. Interestingly, the bundling decision is time-consistent since it can increase the incumbent's profit even after entry occurs in one of these markets.[5] In his paper, products A and B are stand-alone products that are symmetrically identical, but in our paper complementarity between products A and B is asymmetric.

Brandenburger and Nalebuff (1996) point out the prevalence of "co-opetition," in which firms supplying complementary products compete with one another in dividing up industry profits. Bresnahan (1999) and Bresnahan and Greenstein (1999) discuss vertical competition as a salient feature of the computing industry. However, they do not model the phenomenon formally.

We first derive our main results from a simple model with one dimension of preference variation. Section 2 develops the simple model with two firms, A and B. Section 3 solves the pricing game between the two firms by assuming that firm A sets its price before firm B does. Section 4 analyzes the effect of z on firms' profits, consumer surplus, and social welfare. In section 5, we extend the main results/insights in two directions: to the case of simultaneous price-setting, and to the case in which consumers' preferences vary along two dimensions. Concluding remarks follow in the final section.

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[4] In Gilbert and Riordan (2002), if the (exogenously fixed) wholesale price of product A is not sufficiently remunerative, then an incumbent, through technical tying, can design its product A to work better with its own product B than with a rival product B.

[5] In Whinston (1990), Choi and Stefanadis (2001), and Carlton and Waldman (2002), tying is not a profitable choice for an incumbent if entry has already occurred. Tying could deter entry only if an incumbent can commit to tying through technical arrangements.

2. A Model of Product Boundary

There are two firms, A and B, that provide complementary products A and B, respectively. Product A provides some basic functions, and its performance level is measured by a parameter z. Product B by itself does not provide any function, but enhances product A's performance. The combination of products A and B (denoted by (A+B) hereafter) provides a higher performance level $q \geq z$.

If z is equal to zero, then products A and B are strict complements. In contrast, if $z = q$, then product B would be completely superfluous. If $0 < z < q$, then products A and B are asymmetric complements in that product A can function without product B but product B cannot function without product A. The difference (q - z) captures the extent to which product B enhances product A's performance.

Product i's (i=A, B) price and unit production cost are denoted by p_i and c_i, respectively. We assume that firm A sets its price before firm B does. In Section 5.1 we consider an alternative case where the two firms set their prices simultaneously.

Given p_A and p_B, consumers make their purchase decisions. Consumers are assumed to differ with regard to their valuation of product quality. The utility function of a type-θ consumer, $\theta \in [0,1]$, is given by θQ+I, where I is her income spent on numeraire goods, and Q is a quality index of a product. Let the cumulative distribution function and continuous density functions be given by $G(\theta)$ and $g(\theta)$, respectively. Define $F(\theta)$ as the proportion of consumers whose type is higher than θ i.e., $F(\theta)=1-G(\theta)$, and $f(\theta)= -g(\theta)<0$. We make the standard assumption that the distribution of θ satisfies the increasing hazard rate condition: namely, $-f(\theta)/F(\theta)$ is increasing in θ.[6] This assumption yields some key properties of each firm's profit function (see the Appendix).

We impose the following restrictions on key parameters throughout our analysis,

Assumption 1: $c_A + c_B < q$

$$0 \leq z \leq \bar{z} = q - c_B$$

[6] The increasing hazard rate condition is satisfied by most widely used distributions.

The first restriction implies that the maximum willingness to pay for product (A+B) exceeds its unit production cost. Without this restriction, (A+B) will never be supplied. The second restriction implies that the quality enhancement brought by product B (i.e., q-z) exceeds c_B. Without the second restriction, there will be no supply of product B. Under Assumption 1, we have the more interesting case in which both firms A and B are active and the classic double mark-up problem may arise.

Demand Functions for Products A and B

Consumer θ has three options: (i) to buy product A alone and gain net utility $V_A(\theta) = z\theta - p_A$; (ii) to buy (A+B) and gain net utility $V_{A+B}(\theta) = q\theta - p_A - p_B$; (iii) and to buy neither and gain zero net utility.

A necessary condition for the consumer to buy A alone is $\theta \geq \theta_A = p_A/z$. Similarly, a necessary condition for a consumer to buy (A+B) is $q\theta - p_A - p_B \geq 0$, or equivalently $\theta \geq \theta_{A+B} = (p_A + p_B)/q$. Consumers get additional benefits of $(q\text{-}z)\theta$ by purchasing product B in addition to product A. Thus, a necessary condition for a consumer to buy B in addition to A is that $\theta \geq \theta_B = p_B/(q\text{-}z)$. Since $V_A(\theta)$ intersects the steeper function $V_{A+B}(\theta)$ at one point, θ_B, there are only two possible cases.

Case 1. Virtually Independent Products: $\theta_A < \theta_{A+B} < \theta_B$.

This case is illustrated in Figure 1. Consumer types between θ_A and θ_B will buy product A alone, whereas consumer types $\theta \geq \theta_B$ will buy (A+B). That is, consumers with $\theta \geq \theta_A$ will buy product A and consumers with $\theta \geq \theta_B$ will additionally buy product B.

[Figure 1 here]

Substituting the definition of θ_A and θ_B, the demand functions for A and B become:

$$D_A(p_A, p_B) = F(\theta_A) = F\left(\frac{p_A}{z}\right) \quad (1)$$

$$D_B(p_A,p_B) \quad = \quad F(\theta_B) = \quad F(\frac{p_B}{q-z})$$

Since the demand for A depends only on p_A, and the demand for B depends only on p_B, firms A and B act as independent firms so long as their prices satisfy $\theta_A < \theta_{A+B} < \theta_B$. We call this case "virtually independent products."

Case 2: Virtually Strict Complements: $\theta_B < \theta_{A+B} < \theta_A$

Figure 2 illustrates this case. Consumers with $\theta < \theta_{A+B}$ will buy neither products, but consumers with $\theta \geq \theta_{A+B}$ will buy (A+B). None will buy product A alone.

[Figure 2 here]

Substituting the definition of θ_{A+B}, the demand functions for A and B become:

$$D_A(p_A,p_B) = D_B(p_A,p_B) = \quad F(\frac{p_A+p_B}{q}) \qquad (2)$$

Product A provides a quality level z without the aid of B. However, the demand for A and that for B depend on the total price, so their market demand is one of strict complements. We call it the case of "virtually strict complements."

3. The Pricing Game

3. 1 Firm B's Optimal Pricing

Since firms A sets its price before B does, we first examine firm B's optimal price given p_A. The demand function faced by firm B depends on the relative size of θ_B and θ_A.

$$D_B(p_A,p_B) = \begin{cases} D_{B1} = F(\frac{p_B}{q-z}) & \text{if } p_B \geq \frac{q-z}{z} p_A \\ D_{B2} = F(\frac{p_A+p_B}{q}) & \text{if } p_B \leq \frac{q-z}{z} p_A \end{cases}$$

Firm B's demand curve consists of segments of D_{B1} and D_{B2}. Let us define the profit functions corresponding to D_{B1} and D_{B2} as $\Pi_{B1}(p_B; z) = F(\frac{p_B}{q-z})(p_B - c_B)$ and $\Pi_{B2}(p_B; p_A) = F(\frac{p_A + p_B}{q})(p_B - c_B)$, respectively. Firm B maximizes $\Pi_{B1}(p_B; z)$ subject to the constraint $p_B \geq \frac{q-z}{z} p_A$ and $\Pi_{B2}(p_B; p_A)$ subject to the constraint $p_B \leq \frac{q-z}{z} p_A$. Let $p^*_{B1}(z)$ and $p^*_{B2}(p_A)$ denote the unconstrained optimal price of Π_{B1} and Π_{B2}, respectively.[7] As shown in the proof of Lemma 1 in the Appendix, firm B's overall profit function has one peak so that its optimal price is unique. The following lemma shows how firm B's optimal price depends on p_A.

Lemma 1: *Firm B's optimal price depends on p_A and is continuous in p_A. There exist $\underline{p}_A$ and $\overline{p}_A$, where $0 < \underline{p}_A < \overline{p}_A < q - c_B$, such that*

$$p^*_B = \begin{cases} p^*_{B1}(z) & \text{if } p_A \leq \underline{p}_A \\ \frac{q-z}{z} p_A & \text{if } \underline{p}_A \leq p_A \leq \overline{p}_A \\ p^*_{B2}(p_A) & \text{if } \overline{p}_A \leq p_A \leq q - c_B. \end{cases}$$ [8]

Proof: See the Appendix.

It is easy to see that if p_A is zero, then all consumers will get product A. The only question is who will buy product B additionally. From the point of view of firm B, its demand function is given by D_{B1}, and it maximizes profit along D_{B1} by setting its optimal price $p^*_{B1}(z)$, or equivalently by selling its products to consumers whose types are above the cut-off point $\theta^*_B = p^*_{B1}(z)/(q\text{-}z)$. As p_A increases, fewer consumers will buy product A, but as long as the lowest consumer type that buys product A is lower than $p^*_{B1}(z)/(q\text{-}z)$, firm B's optimal price remains unconstrained by p_A.

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[7] Since D_{B1} is independent of p_A, $p^*_{B1}(z)$ depends on z, but not on p_A. In contrast, $p^*_{B2}(p_A)$ depends on p_A, but not on z.

[8] If $p_A > q\text{-}c_B$, then $p^*_B = c_B$, and none will buy product (A+B).

However, once p_A exceeds the threshold $\underline{p}_A$ but remains below $\overline{p}_A$, then the constraint becomes just binding, so firm B's optimal price occurs at the kink $\frac{q-z}{z} p_A$.

If p_A is higher than $\overline{p}_A$, then no consumer is interested in buying only product A alone, and products A and B are always sold together. That is, product B is always sold as a strict complement to A, so B's optimal price is a negative function of p_A.

Figure 3 illustrates firm B's best response to p_A in the case of a uniform distribution.[9] As the figure shows, firm B's best response is not monotonic.

[Figure 3 here]

3. 2 Firm A's Optimal Pricing

Taking firm B's response into account, firm A's reduced-form demand function is as follows:

$$D_A(p_A,\ p_B^*(p_A)) = \begin{cases} D_{A1} = F(\frac{p_A}{z}) & \text{if } p_A < \underline{p}_A \\ D_{A1} = F(\frac{p_A}{z}) & \text{if } \underline{p}_A \le p_A \le \overline{p}_A \\ D_{A2} = F(\frac{p_A + p_{B2}^*(p_A)}{q}) & \text{if } \overline{p}_A \le p_A \end{cases}$$

Combining Lemma 1 with (1) and (2), we see that products A and B are virtually independent if p_A is set below $\underline{p}_A$. If firm A sets p_A above $\overline{p}_A$, then products A and B become virtually strict complements. The intermediate case of $\underline{p}_A \le p_A \le \overline{p}_A$ is right in between the cases of strict complements and independent products. First, products A and B are consumed together, $\theta_A = \theta_B$. Second, after substituting firm B's optimal price $p_B^* = \frac{q-z}{z} p_A$ into equation (2), the resulting demand for firm A is $D_{A1} = F(\frac{p_A}{z})$. That is, firm A perceives its product as independent of product B, but firm B perceives its product as a complement of product A. This case will be called "pseudo complements."

Similar to the case of firm B's demand, firm A's demand curve consists of segments of D_{A1} and D_{A2}.[10] Let us define the profit functions corresponding to D_{A1} and D_{A2} as $\Pi_{A1}(p_A; z) = F(\frac{p_A}{z})(p_A - c_A)$ and $\Pi_{A2}(p_A) = F(\frac{p_A + p^*_{B2}(p_A)}{q})(p_A - c_A)$, respectively. Firm A maximizes $\Pi_{A1}(p_A; z)$ subject to the constraint $p_A \leq \overline{p}_A$ and $\Pi_{A2}(p_A)$ subject to the constraint $p_A \geq \overline{p}_A$. These two profit functions intersect at $\overline{p}_A$. However, as shown in the proof of Lemma 2 in the Appendix, for some range of z, firm A's overall profit function has two peaks rather than one so that firm A's optimal price may not be unique for those z. Let p^*_{A1} (z) and p^*_{A2} denote the unconstrained optimal prices for Π_{A1} and Π_{A2}, respectively.[11] The following lemma shows how firm A's optimal price depends on z.

Lemma 2: Firm A's optimal price is as follows:

$$p^*_A = \begin{cases} p^*_{A2} & \text{if } z \leq z+ \\ p^*_{A1}(z) & \text{if } z \geq z+ , \end{cases} ,$$

where z+ is defined by $\Pi_{A1}(p^*_{A1}; z+) = \Pi_{A2}(p^*_{A2})$, and $0 < z+ < q - c_B$. At z+, firm A's optimal price is either p^*_{A2} or p^*_{A1}(z+), where $p^*_{A1}(z+) < p^*_{A2}$.

Proof: See the Appendix.

Products A and (A+B) are substitutes for one another. Firm A chooses between two different pricing strategies: selling product A as a stand-alone product or selling it as a part of (A+B). If there is no large demand for product A alone, which happens when z is small, then it is optimal for firm A to sell its product as a part of (A+B). However, if a

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[9] In this case, $\underline{p}_A = \frac{z(q - z + c_B)}{2(q - z)}$, $\overline{p}_A = \frac{z(q + c_B)}{2q - z}$, $p^*_{B1} = \frac{q - z + c_B}{2}$, and $p^*_{B2} = \frac{q - p_A + c_B}{2}$.

[10] Firm A's demand is continuous in p_A at $\overline{p}_A$ because $\frac{\overline{p}_A}{z} = \frac{1}{q}[\overline{p}_A + p^*_{B2}(\overline{p}_A)]$.

[11] In the case of a uniform distribution, $p^*_{A1}(z) = \frac{z + c_A}{2}$, and $p^*_{A2} = \frac{q - c_B + c_A}{2}$.

large number of consumers are willing to buy product A alone, which happens when z is large, firm A sells it as an independent product.

Lemma 2 states that p_A^* remains at the same level p_{A2}^* for $0 \leq z \leq z+$, but actually falls from p_{A2}^* to p_{A1}^*(z+) when z crosses z+ from below (see Figure 5). The intuition is as follows. In order to induce consumers to buy product A alone instead of the superior product (A+B), firm A must lower p_A relative to (p_A+ p_B). But this is profitable only if z is sufficiently large relative to q. If z is small, then firm A sells it as a part of product (A+B) and charge a high price p_A , which has the effect of inducing a low p_B.

These considerations behind firm A's optimal pricing strategies are quite similar to those discovered by Gabszewicz and Wauthy (2003). In Gabszewicz and Wauthy's model, there are two vertically differentiated stand-alone products, and a consumer has an option of 'joint purchase' of both products. They analyse the two firms' pricing under the assumption of uniform distribution and zero production costs. They find that "A firm faces two different pricing strategies: either it charges relatively low prices and fights for market shares or it "retreats" with high price on the "rich" side of the market where "joint purchasers" are located." Despite differences in between our model and theirs, in both models firms may face one or two market segments depending on their own prices. [12]

3. 3 Product Relationship

As Figures 1-2 show, the relationships between A and B are characterized by the equilibrium θ_A and θ_B. Since θ_A and θ_B depend on z, let us investigate how the product relationships change with z.

From Lemma 2, we know that for z ∈ [0, z+], firm A sets its price at p_{A2}^*, which is larger than $\overline{p}_A$. Thus, firm B sets its price such that $\theta_B < \theta_A$, and we have the case of

[12] In our model the two product are asymmetric complements. In particular, product B is useless by itself. Because of the difference, an "exclusive purchase equilibrium" (i.e., some consumers buy one product, some other consumers buy the other product, but no consumer buys both products) exists in their model, but not in ours. Also, in our model we have an equilibrium, in which the two firms behave as independent firms. Such an equilibrium does not arise in their model.

virtually strict complements. For z ≥ z+, firm A sets its price as if it is an independent firm. Its optimal price is p^*_{A1}(z), which is lower than $\overline{p}_A$. In order to determine the products' relationship, we need to compare p^*_{A1}(z) and $\underline{p}_A$.

Lemma 3: $p^*_{A1}(z) \leq \underline{p}_A$ if and only if $z \geq \frac{q\,c_A}{c_A+c_B}$.

Proof: See the Appendix.

For $z+ \leq z < \frac{q\,c_A}{c_A+c_B}$, according to Lemma 3, we have $\overline{p}_A > p^*_{A1}(z) > \underline{p}_A$. According to Lemma 1, firm B sets its price such that $p^*_B = \frac{q-z}{z}\, p^*_{A1}$, and we have $\theta_B=\theta_A$, implying that A and B are consumed together. Even though firm A sets its price independently of p_B, firm B behaves as if the two goods are strictly complementary. Thus, we call it the case of "pseudo complements."

For $z \geq \max\{z+, \frac{qc_A}{c_A+c_B}\}$, we have $p^*_{A1}(z) \leq \underline{p}_A$. Firm B sets its price such that $\theta_A < \theta_B$, implying that firm B too behaves as if its demand is independent of p_A. That is, we have the case of virtual independency. The above findings are summarized as follows.

Proposition 1. Let z* denote $\max\{z+, \frac{qc_A}{c_A+c_B}\}$. The relationships between products A and B depend on z in the following ways:

(a) If z ≤ z+, products A and B are virtually strict complements.

(b) If $z+ \leq z < \frac{q\,c_A}{c_A+c_B}$, products A and B are pseudo complements;

(c) If z ≥ z*, products A and B are virtually independent.

If $\frac{q\,c_A}{c_A+c_B} < z+$, then case (b) does not exist.

Figure 4 illustrates Proposition 1.

[Figure 4 here]

4. The Impact of z on Firms, Consumers, and Social Welfare

Usually a dominant firm in one layer extends its product boundary to include functions that are traditionally covered by complementors. In this section, we analyse the impact of z on the firms' prices and profits, consumer surplus, and social welfare for the three cases identified in Proposition 1. In the analysis, we assume that the unit production cost of product A, c_A, does not change with z, which is a good approximation for the software industry.[13] If c_A is allowed to increase with z, our main comparative statics results still hold so long as $\frac{z}{c_A(z)}$ is increasing in z, i.e., the unit cost does not rise faster than z.[14]

4.1 Virtually Strict Complements

From Lemmas 1 and 2, we know that if z is too small, it is not worthwhile for firm A to price its product aggressively to attract consumers who will buy product A alone. Thus, consumers buy only (A+B), and each firm's demand depends on the total price. The firms' equilibrium prices are unaffected by z.[15] Thus, we have

[13] For instance, even though Microsoft expands the functions of Windows by incurring R&D costs, the marginal production cost of a copy of the Windows would remain roughly the same as before.

[14] If $\frac{z}{c_A(z)}$ is increasing in z, then as z goes up, the product relationships change from complementary goods to independent goods, or from complementary goods, to pseudo complementary goods, and to independent goods. Also, the results in Propositions 3-6 in the case of virtual independence and Proposition 7 in the cases of pseudo complements are still valid. But in the case of virtually strict complements, the results are different from those reported in Proposition 2. See footnote 15.

[15] p_A^* increases in firm A's cost. Thus, if c_A increases in z, then p_A^* goes up but p_B^* goes down as z increases.

Proposition 2: If $z \in [0, z+)$, then consumers buy both A and B. The equilibrium prices p_A^* and p_B^* are the same as those for z = 0. The firms' profits, consumer surplus and social welfare are independent of z.

Since A and B are virtually strict complements, the classical double mark-up problem associated with complements is present.

4.2 Virtually Independent Products

Intuitively, one expects that more consumers buy A and fewer consumers buy B as A's own performance z improves. That is indeed the case when A and B are virtually independent, i.e. $z \geq z^*$

The Effect of z on Prices and Profits

Let us examine how the firms' prices change with z when c_A and c_B are positive. For $z \geq z+$, firm A's demand is independent of p_B. Thus, for $z \geq z^*$ (which by definition is at least as large as z+), firm A's optimization problem is to choose p_A^* or equivalently θ_A^*, to maximize

$$\Pi_A = F(\theta_A)(p_A - c_A)$$
$$= zF(\theta_A)(\theta_A - \tfrac{c_A}{z}) \quad (3)$$

Firm A's optimal cut-off point θ_A^* is determined by $\frac{c_A}{z}$ and is an increasing function of $\frac{c_A}{z}$, which can be interpreted as the marginal cost per unit of A's performance. As z increases, θ_A^* becomes smaller, which implies that more consumers buy product A. Proposition 3 below will show that both firm A's profit and p_A^* increase in z.

For $z \geq z^*$, firm B's demand is independent of p_A, and so firm B choose θ_B^* to maximize

$$\Pi_B = F(\theta_B)(p_B - c_B)$$
$$= (q-z)F(\theta_B)(\theta_B - \tfrac{c_B}{q-z}) \quad (4)$$

Since θ_B* is an increasing function of $\frac{c_B}{1-z}$, θ_B^* increases with z, i.e., fewer consumers buy product B as z increases. Proposition 3 will show that both firm B's profit and p_B^* decrease in z.

We can investigate the effect of z on firm B's incentive to increase *q* by analysing the sign of $\frac{\partial^2 \pi_B}{\partial z \partial q}$. As z increases, product A becomes a more attractive substitute for product (A+B), making the positive impact of *q* on firm B's profits smaller, i.e., $\frac{\partial^2 \pi_B}{\partial z \partial q}$ is negative.

Proposition 3:

If c_A and c_B are positive, then in the case of virtually independent products,

(a) p_A^* increases in z, while p_B^* decreases as z increases.

(b) As z increases, more consumers buy product A despite a price increase in p_A^*, and fewer consumers buy product B despite a price decrease in p_B^*.

(c) Firm A's profit increases in z, whereas firm B's profit decreases in z.

(d) An increase in z weakens firm B's incentive to increase *q*.

Proof: See the Appendix.

Double Mark-up Problem

It is well known that two independent monopolies supplying complementary products will set prices that are higher than those set by an integrated monopoly, i.e., the so-called "double mark-up problem." In this section we examine the connection between the pricing behavior of two independent firms A and B and that of a vertically integrated monopolist that supplies both products A and B

The integrated firm that sells products A and B maximizes $\prod^{Int}$ as defined in (5), by choosing two optimal cut-off points, x_A and x_B, where types higher than x_A buy product A, and types higher than x_B buy product B additionally.[16] Since consumers will not buy B without buying A, the firm's choice variables are subject to the constraint $x_A \leq x_B$.

$$\Pi^{Int} = F(x_A)(p_A - c_A) + F(x_B)(p_B - c_B)$$
$$= zF(x_A)(x_A - \tfrac{c_A}{z}) + (q-z)F(x_B)(x_B - \tfrac{c_B}{q-z}) \qquad (5)$$

Obviously, the profit function Π^{Int} is equal to the sum of Π^A and Π^B in (3) and (4). In the absence of the constraint $x_A \leq x_B$, the two parts of the profit function can be maximized independently. If $\frac{c_A}{z} \leq \frac{c_B}{q-z}$, or equivalently if $z \geq \frac{q\, c_A}{c_A + c_B}$, then independent maximization leads to $x_B^* \geq x_A^*$, i.e., the constraint is automatically satisfied. In contrast, if $\frac{c_A}{z} > \frac{c_B}{q-z}$, or equivalently if $z < \frac{qc_A}{c_A + c_B}$, then the constraint becomes binding, implying that the integrated firm will set $x_B^* = x_A^*$. As a result, the profit function depends on the total price. Thus, we obtain

Lemma 4. If $z < \frac{q\, c_A}{c_A + c_B}$, the vertically integrated monopolist behaves as if products A and B are strict complements of one another. If $z \geq \frac{q\, c_A}{c_A + c_B}$, the same firm behaves as if these two products are completely independent.

Using the definition of z^* (namely, $\max\{z+, \frac{qc_A}{c_A + c_B}\}$), we have

Proposition 4: If $z \geq z^*$, then the prices set by independent firms A and B are equal to those set by a vertically-integrated monopolist. That is, the double mark-up problem disappears.

Social Welfare

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[16] The firm can be regarded as maximizing its profit by selling product A and product (A+B). See Johnson and Myatt (2002) for multi-product quality competition. Also, Ellison (2002) analyzes add-on pricing. i.e., the prices of higher quality products are not advertised, while the prices of the basic products are advertised.

Let us investigate the effect of z on social welfare that includes the industry profits and consumer surplus. As Proposition 4 shows, if z ≥ z*, then the sum of firms A and B's profits without cooperation is the same as the vertically integrated monopolist's profits, which unambiguously increases in z. Thus the industry profits increase in z.

Let us look at the effect of z on consumer welfare. The total consumer surplus is given by CS(z) = $\int_{\theta_A(z)}^{\theta_B(z)} (zs - p_A(z))dG(s) + \int_{\theta_B(z)}^{1} (qs - p_A(z) - p_B(z))dG(s)$.

In the case of virtual independent products, θ_B increases in z, but θ_A decreases as z goes up. As z increases, some new consumers buy product A, which adds to total consumer surplus. However, the effect of z on *(p_A+ p_B)* is not clear because while p_A increases in z, p_B decreases in z. Depending on G(θ), *(p_A+ p_B)* can increase in z for some values of z. Thus, for those values of z, an increase in z may reduce the welfare of consumers who continue to buy (A+B), implying that the effect of an increase in z on total consumer surplus is not always positive.

Since the payment by consumers to firms is just a transfer between them, social welfare depends on θ_A and θ_B. As θ_A and θ_B get closer to c_A and (c_A +c_B), respectively, social welfare increases. Thus the effect of an increase in z on social welfare depends on $\frac{\partial \theta_A}{\partial z}$ and $\frac{\partial \theta_B}{\partial z}$. We know that $\frac{\partial \theta_A}{\partial z}$ <0 and $\frac{\partial \theta_B}{\partial z}$ >0. However, for a general distribution G(θ), the combined welfare effect is ambiguous.

However, if θ is uniformly distributed, then the price of (A+B) does not change with z.[17] An increase in z does not affect consumers who continue to buy (A+B), while, by revealed preference, consumers who switch from (A+B) to A alone or from nothing to A alone are better off. Thus, when θ is uniformly distributed, an increase in z has a positive effect on consumer surplus. We summarize the above results as follows.

Proposition 5: In the case of virtually independent products, the industry's total profits increase in z. If the distribution of θ is uniform, then social welfare defined as the sum of industry profits and consumer surplus is increasing in z.

Suppose that firm A can increase z by undertaking a costly R&D project. Let us compare firm A's optimal z and the socially optimal z. To understand the differences between the two, let us identify two effects: a "Business Stealing Effect" and "Spence's quality setting effect." First, firm A increases its profit at the expense of firm B's profit for z>z+. However, firm A does not internalise the negative effect of z on firm B's profit, which makes firm A's optimal z larger than the socially optimal one. Second, when firm A increases its quality, it cares about only the effect on the marginal type, not an average type. Thus, firm A fails to appropriate the increase in consumer surplus, which makes firm A's optimal z smaller than the socially optimal one. The relative size of these two opposing effects is generally ambiguous. However, if $c_A = c_B = 0$, then we can get a clearer welfare result.

The special case of zero production costs

The special case of zero production costs yields some unique welfare results. Products A and B are virtually independent if $z \geq z+$.[18] Under this condition, firm A maximizes $zF(\theta_A)\theta_A$ by choosing θ_A while firm B maximizes $(q\text{-}z)F(\theta_B)\theta_B$ by choosing θ_B, clearly yielding $\theta_A^* = \theta_B^*$. That is, even though the two products are independent of each other, in equilibrium no consumer buys A alone.

Since $p_A^* = z\theta_A^*$ and $p_B^* = (q\text{-}z)\theta_B^* = (q\text{-}z)\theta_A^*$, the total price of (A+B) is $q\theta_A^*$, which is independent of z. It implies that the number of consumers buying (A+B), total industry profits, and social welfare are all independent of z. In other words, z affects only the division of the constant industry profits between the two firms. Thus, when increasing z incurs R&D costs, firm A has too strong R&D incentive compared with social planner. These results are summarized in

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[17] In the case of a uniform distribution, firm A sets its price at $\frac{z+c_A}{2}$ and firm B sets its price at $\frac{q-z+c_B}{2}$. The total price is $\frac{q+c_A+c_B}{2}$, which is independent of z.

[18] When c_A and c_B are zero, we always have $p_{A1}^*(z) \leq \underline{p}_A$, which implies that the region of pseudo complements does not exist. Thus, when c_A and c_B are zero, products A and B are strict complements for $z \leq z+$, and are virtually independent for $z \geq z+$.

Proposition 6: If $c_A = c_B = 0$, then the two products are consumed together even though they are virtually independent. Changes in z beyond z+ do not affect the number of consumers buying (A+B), or the total price of (A+B), or total industry profit, or consumer's surplus. The only effect is changing the division of the same industry profits between the two firms via their prices.

4.3 Pseudo Complements

For the case of pseudo complements, since firm A's demand is independent of p_B, firm A's optimization problem is equivalent to (3). As Proposition 3 shows, p_A^* increases in z, and θ_A^* decreases in z.

The defining characteristics for pseudo complements is that firm B's optimal cut-off point θ_B^* is bound by that of firm A, so that consumers buy both A and B. In this case, as z increases, firm A lowers θ_A^*, allowing firm B to sell its products to more consumers. Since more customers purchase (A+B), the total price ($p_A^* + p_B^*$) must have decreased, which implies that as z increases, p_B^* must have decreased by more than the increase in p_A^*.

Since firm B behaves as if the two products are strict complements, the double mark-up problem persists. However, as z increases, the double mark-up problem becomes less acute, and social welfare increases.

Also as z increases, firm B can sell more quantities of its product, so its marginal profit with respect to q becomes higher, i.e., $\frac{\partial^2 \pi_B}{\partial z \partial q}$ is positive.

Proposition 7:

In the case of pseudo complements,

(a) p_A^* increases in z, while p_B^* decreases in z. As z goes up, ($p_A^* + p_B^*$) decreases and more consumers buy product (A+B).

(b) Firm A's profit increases continuously with z, whereas firm B's profit decreases in z.

(c) Even though the double mark-up problem persists so long as z < $\frac{q\,c_A}{c_A+c_B}$, the problem becomes less acute as z increases from z+ to $\frac{q\,c_A}{c_A+c_B}$, i.e., industry profits, consumer surplus, and social welfare all increase in z.
(d) An increase in z strengthens firm B's incentive to increase *q*.

Proof: See the Appendix.

In both virtually strict and pseudo complements, the double mark-up problem persists. With repeated plays, however, the firms may be able to coordinate their prices to mitigate the problem. If such coordination is difficult to implement or enforce, vertical integration appears to be the only viable alternative left. What we have shown above is that extending firm A's product boundary z beyond a certain level may have a similar efficiency enhancing effect.

Combining Sections 4.1-4.3, we find that if z<z+, then z does not affect the firms' profits, but if z>z+, then firm A's profit continuously increases in z whereas firm B's profit continuously decreases in z. An interesting finding is that while firm A's profit is continuous in z at z+ because $\Pi_{A1}(p^*_{A1}; z+) = \Pi_{A2}(p^*_{A2})$, firm B's profit is discontinuous at z+. Figure 5 illustrates firm A's optimal price, and Figure 6 illustrates the firms' profits, all as functions of z, in the case of a uniform distribution.

[Figure 5 here]

[Figure 6 here]

A higher z means a smaller incremental utility derived from product B (i.e., *q-z*), therefore lowering firm B's profitability. However, as Lemma 2 shows, p^*_A jumps down at z+, so it can be shown that firm B's profit jumps up at z+. That is, since a higher z induces firm A to lower its price, it has a positive effect on firm B's profit. Thus, firm B's profit over a certain range of z > z+ exceeds its profits at z $\in$[0, z+).

5. Extensions and Discussions

5. 1. Simultaneous Price Setting

The above results were derived under the assumption that firms A sets its price before firm B does. In this section we consider the case in which the two firms set their prices simultaneously.

Let us investigate firm A's best response to p_B. When p_B is take as given, firm A maximizes $\Pi_{A1} = F(\frac{p_A}{z})(p_A\text{-}c_A)$ subject to $p_A \leq \frac{z}{q-z} p_B$, but maximizes $\Pi_{A2} = F(\frac{p_A+p_B}{q})$ $(p_A\text{-}c_A)$ subject to $p_A \geq \frac{z}{q-z} p_B$. The two profit functions intersect at $p_A = \frac{z}{q-z} p_B$. Let p^*_{A1}(z) and $p^*_{A2}(p_B)$ denote the unconstrained optimal p_A for the profit functions, Π_{A1} and Π_{A2}, respectively. The following lemma describes firm A's best response.

Lemma 5: *Firm A's optimal price p^*_A depends on p_B. There exist $\tilde{p}_B$* such that

$$p^*_A = \begin{cases} p^*_{A2}(p_B) & \text{if } p_B \leq \tilde{p}_B \\ p^*_{A1}(\text{z}) & \text{if } p_B \geq \tilde{p}_B, \end{cases}$$

where $p^*_{A2}(\tilde{p}_B) > p^*_{A1}(z)$, so firm A's best response function is not continuous at $\tilde{p}_B$.

Proof: See the Appendix.

If p_B is low, firm A sells its product as a part of (A+B). However, if p_B is high, firm A sells its product as an independent product. Again, this reflects the same insight as discussed under Lemma 2: it is worthwhile for firm A to induce low-type consumers to buy product A alone by setting its price low if the market for (A+B) is limited by a high price of the complementor.

As lemma 5 shows, firm A's best reponse is not continuous because its overall profit function has two peaks over some range of p_B. When firm B sets its price at $\tilde{p}_B$, firm A has two optimal prices: $p^*_{A2}(\tilde{p}_B)$ and p^*_{A1}(z).

Nash Equilibrium with Simultaneous Price Setting.

The Nash equilibrium of the simultaneous price-setting game is given by the intersection of firm A's best response function as summarized in Lemma 4 and that of firm B as summarized in Lemma 1. Firm B's best respone is continuous because its profit function has a single peak, but firm A's best response function is not continuous. As a result, the existence of a pure-strategy Nash equilibrium is not guaranteed but depends on the level of z. If θ is uniformly distributed, the following Proposition provides a precise characterization of the Nash equilibrium.

Proposition 8: In the case of uniform distribution, there exist two critical values of z: z_{min}, z_{max}, with $z_{min} < z_{max}$ such that the following holds:

(a) If $z < z_{min}$, there exist a unique pure-strategy Nash equilibrium which yields virtually strict complements.

(b) If $z > z_{max}$, there exists a unique pure-strategy Nash equilibrium which yields virtually independent products.

(c) If $z_{min} < z < z_{max}$, no pure strategy Nash equilibrium exists, but a mixed strategy equilibrium exists. In the mixed strategy equilibrium, with certain probability the products are virtually strict complements and otherwise they are virtually independent.

Proof: See the Appendix.

The existence and non-existence of a pure-strategy Nash equilibrium are illustrated in Figures 7.

[Figure 7 here]

If $z > z_{max}$, as illustrated in figure 7-(a), in equilibrium the firms behave as if they were two independent firms; if $z < z_{min}$, as illustrated in figure 7-(c), in equilibrium the firms behave as if their products were strict complements.

The nature of a mixed-strategy Nash equilibrium is illustrated in Figure 7-(b). If firm B sets its price at $\tilde{p}_B$, firm A is indifferent between p_{A1} and p_{A2}, and would therefore be indifferent between any randomization of those two prices. There exist probabilities α for p_{A1} and (1-α) for p_{A2} such that firm B's best response to firm A's randomization is exactly $\tilde{p}_B$. In the mixed strategy equilibrium, with probability α the products are independent and with probability (1-α) the products are strict complements. That is to say, the case of pseudo complements is a consequence of the sequential price setting.

The following figures shows how p_A and p_B change with z when a pure-strategy Nash equilibrium exists for the case in which q=1 and $c_a = c_b = 0.15$. For this case, we have $z_{max} \approx 0.59$ and $z_{min} \approx 0.48$

[Figure 8 here]

5.2. More General Demand Specification: Preference Diversity along Two Dimensions

In Sections 2-4, a consumer's valuation of products A and (A+B) depends on only on a single parameter θ, given z and q. Thus, for each firm, the marginal consumer type is uniquely determined either by its own price alone or by the sum of the two prices ($p_A + p_B$).[19] As a result, each firm sets its price as if the two products are independent or are strict complements.[20] To check the robustness of the results obtained above, we consider a more general demand specification: consumers' preference toward products A and B vary along two dimensions.

Suppose that product A provides its own independent stand-alone value, which is denoted by v. Consumers get $U_A(\theta, v) = v + z\theta$ from product A alone. By purchasing product B additionally, consumers get $U_{A+B}(\theta, v) = v + q\theta$ from product A+B. That is,

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[19] The marginal consumer type for a product is the lowest type among consumers buying the product. For instance, when $\theta_A < \theta_B$, firm A's marginal type is θ_A, which is $\frac{p_A}{z}$; when $\theta_A > \theta_B$, firm A's marginal type is θ_{A+B}, which is ($p_A + p_B$).

[20] We would like to thank an anonymous referee for providing us with this insight and for making several usefule suggestions in the analysis of the two-demension case.

product B enhances the quality of the second component from z to 1. For simplicity, we set q=1. We assume that consumer preferences vary in two dimensions: θ and v.[21] In the remainder of this section, (θ, v) are distributed independently over [0, 1] x [0, $\bar{v}$]. The situation is as depicted in Figure 9.

[Figure 9 here]

The vertical axis measures v, the consumers' value for the stand-alone function, and the horizontal axis measures θ as before. For a consumer to buy product A alone, we must have $V_A(\theta, v) = U_A (\theta, v) - p_A \geq 0$, as shown in Figure 9 (a). Similarly, a necessary condition for a consumer to buy (A+B) is that $V_{A+B} (\theta, v) = U_{A+B} - p_A - p_B \geq 0$, as shown by Figure 9 (b). A necessary condition for a consumer to buy product B in addition to product A is that $\theta \geq \theta_B = p_B/(1\text{-}z)$ as represented by the rectangle on the right end of Figure 9 (c).[22]

Demand Functions for Products A and B

Even though the analysis becomes more complicated when consumer preferences vary along two dimensions, the cases of virtually strict complements and virtually independent products may still arise under certain parametric values. Similar to Section 2, the intersection of V_A and V_{A+B} plays an important role in determining the demand functions for products A and B. The intersection of V_A and V_{A+B} occurs at ($\theta_B = p_B/(1\text{-}z)$ and $v = p_A - \frac{z}{1-z} p_B$.) As it turns out, the nature of the demand functions for A and B

[21] If v is a constant for all consumers, then V_A (θ) =zθ-(p_A- v) and V_{A+B} (θ) =qθ-(p_A-v+p_B). The analysis in Sections 2-4 applies to this new demand specification if p_A is replaced by (p_A-v), and the earlier results remain intact.

[22] For any quantity of B to be demanded at all, p_B must be less than ($1\text{-}z$) as in the basic model.

depends on whether the critical value of this intersection occurs above, below, or within $[0, \bar{v}]$.[23] Figure 10 shows the three different cases.

[Figure 10 here]

If $(p_A - \frac{z}{1-z} p_B) > \bar{v}$, the situation is as depicted in Figure 10 (a).[24] Consumers above V_{A+B} will buy products (A+B). The demand for either product depends on the total price (p_A+p_B), so the two firms behave as if the two products are strict complements. The level of z does not affect the firms' pricing and profits.

If $(p_A - \frac{z}{1-z} p_B) < 0$, the situation is as depicted in Figure 10 (b).[25] Consumers above V_A buy product A, and consumers to the right of θ_B buy product B additionally. Demand for the two products becomes independent: the demand for B (the dark area) depends only on θ_B, which is independent of p_A; the demand for A (light area + dark area) comes from consumers who buy A alone and those who buy A together with B, but the total demand for A depends on p_A alone. Thus, in this case each product's demand depends only on its own price, and there is no double mark-up problem.

If $\bar{v} \geq (p_A - \frac{z}{1-z} p_B) \geq 0$, then the situation is as depicted in Figure 10 (c). Demand for B is given by the dark area, whereas demand for A is given by both the light and dark areas. In this case, the demand for (A+B) differs from that of Figure 11(b) due to the triangle JKL (or a trapezoid if $(p_A+p_B)>1$). Since the hypotenuse of the triangle (which is the locus of firm B's marginal consumer types) is a function of (p_A+p_B) and the vertical line depends on p_B/z, the demand for product B is a function of (p_A+p_B) as well as p_B. Moreover, firm A's marginal consumer types include those of firm B as well as those

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[23] Depending on the inequalities between (i) p_A/z and $p_B/(1-z)$, (ii) (p_A+p_B) and 1, (iii) $p_A - \frac{z}{1-z} p_B$ and $\bar{v}$, and (iv) p_A and $\bar{v}$, we have different demand systems. However, these demand systems can be categorized into the three regimes depending on the inequality between $(p_A - \frac{z}{1-z} p_B)$ and $\bar{v}$.

[24] The inequality $p_A - \frac{z}{1-z} p_B > \bar{v}$ implies that $p_A > \bar{v}$.

[25] The inequality implies that $\frac{p_A}{z} < \frac{p_B}{1-z}$. Also, since $\frac{p_B}{1-z}$ is less than 1, the inequality $\frac{p_A}{z} < \frac{p_B}{1-z}$ implies that $\frac{p_A}{z} < 1$.

who lie on the locus of $V_A(\theta, v) = 0$ to the left of point J (which depends on p_A). Thus, demand for product A depends on (p_A+p_B) as well as p_A.

In Sections 2-4 where consumers differ in only one dimension, θ, each firm's marginal consumer is determined either only by its own price or only by (p_A+p_B). However, in the two-dimensional case, this simple dichotomy is lost, and each firm may have a continuum of marginal consumers, which depends on both its own price as well as the total price. Therefore, we refer to the situation depicted in Figure 10 (c) as an intermediate case of "mixed demand."

Optimal Pricing in a Numerical Simulation

Let us obtain the firms' optimal prices *numerically* under the assumption that firm A sets its price before firm B does.[26] Under the assumption that (θ, *v*) are uniformly distributed [0, 1] x [0, $\bar{v}$], we analyse how the firms' optimal prices change with z and $\bar{v}$.

As before, firm A may choose between two different pricing strategies: selling product A as a stand-alone product or selling it as a part of (A+B). A large number of consumers will buy A alone either if z is large, or if $\bar{v}$ is large, or both. Thus, firm A's optimal pricing strategy depends on both z and $\bar{v}$. The numerical results show that the effects of z on the firms' equilibrium prices critically depend on the size of $\bar{v}$. Figure 11 shows the effects of z on p_A, p_B and (p_A+ p_B) when $\bar{v}$ is 0.4.

[Figure 11 here]

The figure shows that as z increases, the relationship between products A and B changes from the regime of virtually strict complements to the intermediate mixed regime, and then to the regime of independent products. As before, when z is small, it is too costly for firm A to cut p_A to attract some consumers to buy A alone. Thus, firm A sets its price so

[26] We also numerically analyse the case in which the two firms set their prices simultaneously. The Nash equilibrium outcomes in that case are quite similar to those of this section: If $\bar{v}$ is large enough, we have only two demand regimes (independent products and mixed demand); if $\bar{v}$ is small, we have all the three demand regimes, including that of strict complements. Furthermore, if $\bar{v}$ is small, then for some ranges of z there is no pure-strategy equilibrium, which is similar to the results in Proposition 8, and if $\bar{v}$ is large enough, pure-strategy Nash equilibrium exists for all z.

that the two products are consumed together, and small charges in z have no effect on each firm's optimal pricing. However, if z is sufficiently large, then firm A will find it profitable to set its price to attract low θ consumers to buy product A alone. For some large values of z, firm A sets its price as if its product were independent of product B.

Figure 12 shows the case in which $\bar{v}$ is large ($\bar{v}$ = 1). In this case, the regime of virtually strict complements disappears altogether. Since $\bar{v}$ is large, there are enough number of consumers who are willing to buy product A alone even if z is low, and we have the intermediate case of mixed demand for small z. As z increases, product B becomes less important as a complement of product A, so firm A sets its price as if its product were independent of that of product B.

[Figure 12 here]

Figure 13 below shows how the demand regimes (complements, mixed, and independent products) depend on z and $\bar{v}$. As the figure shows, there is no pseudo-complementarity regime in the two dimensional case.

[Figure 13 here]

Comments on the distribution of (θ, v)

(1) The distribution of consumer preferences in the two-dimensional case analysed above is equivalent to that over the space spanned by the product of $\bar{v}$ and [0, 1]. The parameter $\bar{v}$ can be interpreted as the quality level of the stand-alone function, and the consumers' valuation of the quality is distributed over [0, 1].

Our numerical results were obtained under the assumption that θ and *v* were distributed over $[0,1]\mathrm{x}[0,\bar{v}]$ independently and uniformly. However, in reality consumers with high *v* tend to have high θ, i.e., θ and *v* tend to be positively correlated. The results obtained from our basic model in Sections 2-4 can be applied to the case in which θ and v are correlated.

(2) If θ and v are linearly related, say, $v = s\theta$, then, the two-dimensional model is structurally identical to our basic model because $V_A(\theta,v)$ becomes $(\tilde{z}\theta - p_A)$ and $V_{A+B}(\theta,v)$ becomes $(\tilde{q}\theta - p_A - p_B)$, where $\tilde{z}$ = z+s and $\tilde{q} = q+s$.

(3) Suppose that $v = s\theta + \varepsilon$, where ε is uniformly distributed between $[0, \bar{\varepsilon}]$. Then $V_A(\theta,v) = (z+s)\theta - p_A + \varepsilon$, and $V_{A+B}(\theta,v) = (q+s)\theta - p_A - p_B + \varepsilon$, or equivalently $V_A(\theta) = \tilde{z}\theta - p_A + \varepsilon$, and $V_{A+B}(\theta) = \tilde{q}\theta - p_A - p_B + \varepsilon$. It is exactly equivalent to the two-dimensional case analysed above in this section, so our results apply directly: If the dispersion of consumer types around the mean as captured by $\bar{\varepsilon}$ is small, then we may have all of the three demand regimes; if $\bar{\varepsilon}$ is large, then we have only the case of mixed demand and the case of virtually independent products.

6. Conclusions

A dominant firm in one layer of a multi-layered system often seeks to extend the functions of its products to include functions that are traditionally provided by firms in other layers. The definition of product boundaries changes continuously as a consequence of vertical competition. In this paper, we investigate the strategic interaction between two “adjacent” firms when product A is a required complement for product B, and product B enhances the performance of product A. We also analyze how firm A’s product boundary expansion affects pricing, profits, social welfare, and the other firm’s R&D incentive.

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Appendix

The increasing hazard rate condition yields strictly quasi-concave profit functions.

Consider a monopolist supplying a product of quality level z . Without any loss of generality, assume that the marginal cost of the product is zero. Let *p* denote the price of the product and θ denote the firm's cutoff point i.e., the lowest consumer type buying the product. By definition, $\theta = p/z$. Since there is a one-to-one mapping between *p* and θ, the firm maximizes $z\theta F(\theta)$ by choosing its optimal cutoff point θ. The first-order condition of profit maximization is given by $zF(\theta)+z\theta f(\theta) = 0$. The second derivative of the profit function is $2zf(\theta)+z\theta f'(\theta)$, which upon substitution of the first-order condition becomes $2zf(\theta)-zF(\theta)f'(\theta)/f(\theta)$.

The increasing hazard rate condition (i.e., $-f/F$ is increasing) implies $(f(\theta))^2-F(\theta)(f'(\theta))>0$, which in turn implies $2zf(\theta)-zF(\theta)f'(\theta)/f(\theta)<0$ because $f(\theta)<0$. That is to say, the profit function is locally strictly concave in θ whenever its slope is zero, i.e., the function is strictly quasi-concave in θ. Q.E.D

Proof of Lemma 1

Firm B maximizes $\Pi_{B1} = F(\frac{p_B}{q-z})(p_B-c_B)$ subject to $p_B \geq \frac{q-z}{z} p_A$ but maximizes $\Pi_{B2} = F(\frac{p_A+p_B}{q})(p_B-c_B)$ subject to $p_B \leq \frac{q-z}{z} p_A$. Under the increasing hazard rate condition, each of these two functions is strictly quasi-concave in p_B and has a single peak. Let p^*_{B1} and p^*_{B2} denote the unconstrained optimal p_B for the profit functions, $\Pi_{B1}(p_B)$ and $\Pi_{B2}(p_B)$, respectively. The first-order conditions of these profit functions are as follows:

$$\frac{\partial \Pi_{B1}}{\partial p_B} = F(\frac{p_B}{q-z}) + f(\frac{p_B}{q-z})(p_B-c_B)\frac{1}{q-z} = 0$$

$$\frac{\partial \Pi_{B2}}{\partial p_B} = F(\frac{p_A+p_B}{q}) + f(\frac{p_A+p_B}{q})(p_B-c_B)\frac{1}{q} = 0$$

The two first-order conditions evaluated at the kink $p_B = \frac{q-z}{z} p_A$ become

$$p_A + \frac{F(\frac{p_A}{z})}{f(\frac{p_A}{z})} z = \frac{z}{q-z} c_B \qquad \text{(A1)}$$

$$p_A + \frac{F(\frac{p_A}{z})}{f(\frac{p_A}{z})}\frac{qz}{q-z} = \frac{z}{q-z}c_B, \qquad \text{(A2)}$$

Since $F(\theta)/f(\theta)$ is increasing in θ, the left-hand side of (A1) and (A2) is monotonically increasing in p_A. Thus, given q, z, and c_B, there is a unique p_A that solves (A1) or (A2), respectively. Denote the unique p_A satisfying equations (A1) and (A2) by $\underline{p}_A$ and $\overline{p}_A$, respectively. It can be shown that $0 < \underline{p}_A < \overline{p}_A < q - c_B$.

There are three cases:

Case1 ($p_A \le \underline{p}_A$): When evaluated at $p_B = \frac{q-z}{z} p_A$, we have $\frac{\partial \Pi_{B1}}{\partial p_B} \ge 0$ and $\frac{\partial \Pi_{B2}}{\partial p_B} > 0$ because f(.)<0. Since $\frac{\partial \Pi_{B1}}{\partial p_B} \ge 0$, p^*_{B1} that maximizes Π_{B1} does not violate the constraint that $p_B \ge \frac{q-z}{z} p_A$. However, p^*_{B2} that maximizes Π_{B2} violates the constraint that $p_B \le \frac{q-z}{z} p_A$. Because $\Pi_{B1}(\frac{q-z}{z} p_A) = \Pi_{B2}(\frac{q-z}{z} p_A)$, the overall profit function has a single peak at p^*_{B1}, and firm B's global optimum price is given by p^*_{B1}.

Case 2 ($\overline{p}_A \le p_A$): In this case, when evaluated at $p_B = \frac{q-z}{z} p_A$, we have $\frac{\partial \Pi_{B1}}{\partial p_B} < 0$ and $\frac{\partial \Pi_{B2}}{\partial p_B} \le 0$. An argument similar to that for Case 1 establishes that the global optimum price is given by p^*_{B2}.

Case 3 ($\underline{p}_A < p_A < \overline{p}_A$): In this case, when evaluated at $p_B = \frac{q-z}{z} p_A$, we have $\frac{\partial \Pi_{B1}}{\partial p_B} < 0$ and $\frac{\partial \Pi_{B2}}{\partial p_B} > 0$, so the global maximum is achieved at the kink, i.e., $p^*_B = \frac{q-z}{z} p_A$.

Using the definition of $\underline{p}_A$ and $\overline{p}_A$, the strict inequalities of Case 3 can be extended to become weak inequalities, and thus firm B's optimal price is continuous in p_A.

Q. E. D

Proof of Lemma 2:

Let us define the profit functions corresponding to D_{A1} and D_{A2} as $\Pi_{A1}(p_A; z) = F(\frac{p_A}{z})(p_A - c_A)$ and $\Pi_{A2}(p_A) = F(\frac{p_A + p^*_{B2}(p_A)}{q})(p_A - c_A)$, respectively. Let p^*_{A1} and p^*_{A2} denote their respective unconstrained optimal prices. The first-order condition of these profit functions are as follows:

$$\frac{\partial \Pi_{A1}}{\partial p_A} = F(\tfrac{p_A}{z}) + f(\tfrac{p_A}{z})\,(p_A - c_A)\,\tfrac{1}{z} = 0$$

$$\frac{\partial \Pi_{A2}}{\partial p_A} = F(\tfrac{p_A + p_B}{q}) + f(\tfrac{p_A + p_B}{q})\,(1 + \tfrac{dp_B}{dp_A})(p_A - c_A)\,\tfrac{1}{q} = 0$$

Evaluating these two first-order conditions at the kink $\bar{p}_A$ and taking into account the fact that firm B sets its price equal to $\frac{q-z}{z}\bar{p}_A$ when p_A is $\bar{p}_A$, the two first-order conditions evaluated at $\bar{p}_A$ become

$$\bar{p}_A - c_A = -z\frac{F(\frac{\bar{p}_A}{z})}{f(\frac{\bar{p}_A}{z})} \qquad \text{(A3)}$$

$$(1 + \tfrac{dp_B}{dp_A})(\bar{p}_A - c_A) = -q\frac{F(\frac{\bar{p}_A}{z})}{f(\frac{\bar{p}_A}{z})} \qquad \text{(A4)}$$

Since $-F(\theta)/f(\theta)$ is decreasing in θ, the right-hand side of (A3) and (A4) is monotonically increasing in z. Thus, there exist unique z_1 and z_2 that solve (A3) and (A4), respectively. Moreover, using the facts that $\frac{dp_B}{dp_A}<0$, $q>z$, and $f(\theta)<0$ in (A3) and (A4), it can be shown that $z_1 < z_2$. As in the proof of Lemma 1, there are three different cases:

Case1 ($z \le z_1$): When evaluated at $\bar{p}_A$, we have $\frac{\partial \Pi_{A1}}{\partial p_A} \ge 0$ and $\frac{\partial \Pi_{A2}}{\partial p_A} > 0$. It implies that p^*_{A2} that maximizes Π_{A2} does not violate the constraint that $p_A \ge \bar{p}_A$. However, p^*_{A1} that maximizes Π_{A1} is equal to or larger than $\bar{p}_A$. Since $\Pi_{A1}(\bar{p}_A; z) = \Pi_{A2}(\bar{p}_A)$ for all z, firm A's global optimum is given by the peak of Π_{A2} and is achieved at p^*_{A2}, which is larger than $\bar{p}_A$.

Case2 ($z_2 \le z$): Again evaluated at $\overline{p}_A$, $\frac{\partial \Pi_{A1}}{\partial p_A}$ <0 and $\frac{\partial \Pi_{A2}}{\partial p_A}$ ≤ 0. An argument similar to that for Case 1 establishes that firm A's globally optimal price is p^*_{A1} , which is less than $\overline{p}_A$.

Case3 ($z_1 < z < z_2$): Evaluated at $\overline{p}_A$, we have $\frac{\partial \Pi_{A1}}{\partial p_A}$ <0 and $\frac{\partial \Pi_{A2}}{\partial p_A}$ >0. It implies that the overall profit function has two peaks, one at p^*_{A1} and the other at p^*_{A2}, where $p^*_{A1} < \overline{p}_A < p^*_{A2}$. Thus we need to compare $\Pi_{A1}(p^*_{A1};z)$ and $\Pi_{A2}(p^*_{A2})$ for $z \in (z_1, z_2)$ to ascertain firm A's globally optimal price, bearing in mind that $\Pi_{A1}(p^*_{A1};z)$ is continuously increasing in z while $\Pi_{A2}(p^*_{A2})$ is independent of z.

From the definition of z_1, $\Pi_{A1}(p_{A1};z_1)$ is maximized at $\overline{p}_A$. Thus, $\Pi_{A1}(p^*_{A1};z_1) = \Pi_{A1}(\overline{p}_A;z_1) = \Pi_{A2}(\overline{p}_A) < \Pi_{A2}(p^*_{A2})$, where the second equality holds because $D_{A1}(\overline{p}_A) = D_{A2}(\overline{p}_A)$. From the definition of z_2, $\Pi_{A2}(p_{A2})$ is maximized at $\overline{p}_A$. Thus, $\Pi_{A2}(p^*_{A2}) = \Pi_{A2}(\overline{p}_A) = \Pi_{A1}(\overline{p}_A;z_2) < \Pi_{A1}(p^*_{A1};z_2)$, where the second equality holds because $D_{A1}(\overline{p}_A) = D_{A2}(\overline{p}_A)$.

Combining the above inequalities yields $\Pi_{A1}(p^*_{A1};z_1) < \Pi_{A2}(p^*_{A2}) < \Pi_{A1}(p^*_{A1};z_2)$. Since $\Pi_{A1}(p^*_{A1};z)$ is continuously increasing in z, and $\Pi_{A2}(p^*_{A2})$ is independent of z, there exists a unique z+ strictly between z_1 and z_2 such that $\Pi_{A1}(p^*_{A1};z+) = \Pi_{A2}(p^*_{A2})$; for $z < z+$, we have $\Pi_{A1}(p^*_{A1};z) < \Pi_{A2}(p^*_{A2})$; for $z > z+$, we have $\Pi_{A1}(p^*_{A1};z) > \Pi_{A2}(p^*_{A2})$.

These inequalities also hold if z takes on extreme values: if z= 0, then $\Pi_{A1}(p^*_{A1};z) = 0 < \Pi_{A2}(p^*_{A2})$; if $z \ge q - c_B$, then the demand for product (A+B) is zero, and $\Pi_{A1}(p^*_{A1};z) > \Pi_{A2}(p^*_{A2}) = 0$. Thus, $0 < z^+ < q - c_B$.

Since z+ lies between z_1 and z_2, by combining Cases 1-3, we prove that firm A's global optimum is achieved at p^*_{A1} if $z \ge z+$ and is achieved at p^*_{A2} if $z \le z+$. Finally, the inequality that $p^*_{A1}(z+) < p^*_{A2}$ follows from $p^*_{A1}(z+) < \overline{p}_A < p^*_{A2}$. Q. E. D

Proof of Lemma 3

By definition, $\underline{p}_A$ satisfies (A1), and p^*_{A1} satisfies (A3). By comparing (A1) and (A3), we find that $p^*_{A1} < \underline{p}_A$ if and only if z > $\frac{q\, c_A}{c_A + c_B}$. Q. E. D

Proof of Proposition 3

Part (a):

Firm A's optimal price satisfies the first-order condition of (3), i.e.,

$$p^*_A = c_A - z\frac{F(\theta_A)}{f(\theta_A)}$$

Since $(-\frac{F(\theta_A)}{f(\theta_A)})$ is decreasing in θ_A and θ_A is decreasing in z, the right-hand side of the above condition is increasing in z, i.e., p^*_{A1}(z) is increasing in z for all z ≥ z+.

Firm B's optimal price satisfies the following first-order condition of (4) i.e.,

$$p^*_B = c_B - \frac{F(\theta_B)}{f(\theta_B)}(q - z)$$

Since $(-\frac{F(\theta_B)}{f(\theta_B)})$ is decreasing in θ_B and θ_B is increasing in z, the right-hand side of the above condition is decreasing in *z*. Thus, p^*_{B1} (z) decreases in z for all z ≥ z*.

Part (b) follows from changes in θ^*_A and θ^*_B.

Part (c) : As firm A maximizes $zF(\theta_A)(\theta_A - \frac{c_A}{z})$, its profit clearly increases in z for z> z+.

As firm B maximizes $(q-z)F(\theta_B)(\theta_B - \frac{c_B}{q-z})$, its profit deceases in z for z>z*. Q.E.D.

Part (d): Firm B's profit function is $F(\theta_B)((q-z)\theta_B - c_B)$, implying that the important variable is the size of $k = (q-z)$, not the individual values of *q* and z.

We show that $\Pi_B(.)$ is convex in *k*, i.e., $\lambda\Pi_B(k_1) + (1-\lambda)\Pi_B(k_2) \geq \Pi_B(\lambda k_1 + (1-\lambda)k_2)$. Let θ_1, θ_2 and θ_3 denote the optimal cutoff points for k_1, k_2 and $\lambda k_1 + (1-\lambda)k_2$, respectively.

$$\Pi_B(\lambda k_1+(1-\lambda)k_2)$$
$$=F(\theta_3)((\lambda k_1+(1-\lambda)k_2)\theta_3-c_B)$$
$$=\lambda F(\theta_3)(k_1\theta_3-c_B)+(1-\lambda)F(\theta_3)(k_2\theta_3-c_B)$$
$$\leq\lambda F(\theta_1)(k_1\theta_1-c_B)+(1-\lambda)F(\theta_2)(k_2\theta_2-c_B)$$
$$=\lambda\Pi_B(k_1)+(1-\lambda)\Pi_B(k_2),$$

where the inequality follows from the definition that θ_1 is the optimal cut-off point under k_1 and θ_2 is the optimal cut-off point under k_2.

Since $\Pi_B(.)$ is convex with respect to (q-z), $\frac{\partial^2\Pi_B}{\partial z\partial q}$ is negative. Q.E.D

Proof of Proposition 7

Part (a): For z > z+, firm A maximizes (3) and its optimal price is given by p^*_{A1}(z). As shown in Proposition 3, p^*_A is increasing in z, and θ^*_A is decreasing in z. Since $\theta^*_A = \theta^*_B$, more consumers buy (A+B) as z increases, which implies that (p_A+p_B) and p^*_B falls as z increases.

Part (b): As firm A maximizes $zF(\theta_A)(\theta_A-\frac{c_A}{z})$, its profit clearly increases in z for z> z+. However, for z∈ (z+, $\frac{qc_A}{c_A+c_B}$], $\theta^*_A = \theta^*_B$, and firm B's profit function becomes $\Pi_B(p_B) = F(\frac{p_A+p_B}{q})(p_B - c_B)$, which is decreasing in p_A. Since p_A increases in z, it follows that firm B's profit decreases in z between z+ and $\frac{c_A}{c_A+c_B}$.

Part (c): If z lies between z+ and $\frac{qc_A}{c_A+c_B}$, the vertically integrated firm maximizes $qF(x_{A+B})(x_{A+B}-\frac{c_A+c_B}{q})$ by choosing x^*_{A+B}, and x^*_{A+B} is determined by $\frac{c_A+c_B}{q}$. However, firm A maximizes $zF(\theta_A)(\theta_A-\frac{c_A}{z})$ by choosing θ^*_A, and θ^*_A is determined by $\frac{c_A}{z}$. The inequality $z<\frac{q\,c_A}{c_A+c_B}$ is equivalent to $\frac{c_A+c_B}{q}<\frac{c_A}{z}$, which implies $x^*_{A+B}<\theta^*_A$. That is, θ^*_A is too high in terms of A and B's joint profits, i.e., the double mark-up problem persists.

However, as z increases toward $\frac{qc_A}{c_A+c_B}$, θ^*_A becomes closer to x^*_{A+B}, so that industry profits become closer to the integrated monopoly's profits.

Products A and B are consumed together, and thus the consumer surplus depends on only (p_A+p_B). As Part (a) shows, the total price decreases in z. Thus, industry profits, consumer surplus, and social welfare all increase in z.

Part (d) Firm A sets its price at p^*_{A1}, and firm B sets its price such that $\theta_B = \theta_A$. Thus, firm B's profit function becomes, $\Pi_B = F(\theta_A)(p_B - c_B) = F(\theta_A)((q\text{-}z)\theta_A - c_B)$. Since $\frac{\partial \Pi_B}{\partial q}$ is $F(\theta_A)\theta_A$, $\frac{\partial^2 \Pi_B}{\partial z \partial q} = [f(\theta_A)\theta_A + F(\theta_A)]\frac{\partial \theta_A}{\partial z}$. Firm A's price satisfies the following first-order condition, $F(\frac{p_A}{z}) + f(\frac{p_A}{z})(p_A\text{-}c_A)\frac{1}{z} = 0$, which implies that $F(\theta_A) + f(\theta_A)\,\theta_A = f(\theta_A)c_A\frac{1}{z} < 0$. Since $\frac{\partial \theta_A}{\partial z} < 0$, we have $\frac{\partial^2 \Pi_B}{\partial z \partial q} = [f(\theta_A)\theta_A + F(\theta_A)]\frac{\partial \theta_A}{\partial z} > 0$.

Q.E.D.

Proof of Lemma 5

Firm A's profit function is composed of the two underlying profit functions: $\Pi_{A1} = F(\frac{p_A}{z})(p_A\text{-}c_A)$ and $\Pi_{A2} = F(\frac{p_A+p_B}{q})(p_A\text{-}c_A)$. The kink of product A's demand curve occurs at $p_A = \frac{z}{q-z}p_B$. Let p^*_{A1} and p^*_{A2} be the unconstrained optimal prices corresponding to Π_{A1} and Π_{A2}, respectively.

Evaluating the two first-order conditions of Π_{A1} and Π_{A2}, respectively, at the kink and rearranging yields

$$p_B + (q\text{-}z)\frac{F(\frac{p_B}{q-z})}{f(\frac{p_B}{q-z})} = \frac{q-z}{z}c_A \qquad \text{(A5)}$$

$$p_B + q\frac{q-z}{z}\frac{F(\frac{p_B}{q-z})}{f(\frac{p_B}{q-z})} = \frac{q-z}{z}c_A \qquad \text{(A6)}$$

Since $F(\theta)/f(\theta)$ is increasing in θ, the left-hand side of (A5) and (A6) is monotonically increasing in p_B. Thus, there exist unique $\underline{p}_B$ and $\overline{p}_B$ that solve (A5) and (A6), respectively. Since f(θ) is negative and q>z, we have $0< \underline{p}_B < \overline{p}_B$ and there are three cases:

Case1 ($p_B \leq \underline{p}_B$): When evaluated at the kink $p_A = \frac{z}{q-z} p_B$, we have $\frac{\partial \Pi_{A1}}{\partial p_A} \geq 0$ and $\frac{\partial \Pi_{A2}}{\partial p_A} >$ 0. It implies that p^*_{A1} that maximizes Π_{A1} is equal to or larger than $\frac{z}{q-z} p_B$, while p^*_{A2} that maximizes Π_{A2} does not violates the condition $p_A \geq \frac{z}{q-z} p_B$. Because $\Pi_{A1}(\frac{z}{q-z} p_B) = \Pi_{A2}(\frac{z}{q-z} p_B)$, firm A's global optimum is given by the peak of Π_{A2} at p^*_{A2}.

Case 2 ($\overline{p}_B \leq p_B$) : Again evaluated at the kink, $\frac{\partial \Pi_{A1}}{\partial p_A}$ <0 and $\frac{\partial \Pi_{A2}}{\partial p_A} \leq 0$. An argument similar to that for case 1 establishes that firm A's globally optimal price is p^*_{A1}.

Case 3 ($\underline{p}_B \leq p_B \leq \overline{p}_B$): Because $\frac{\partial \Pi_{A1}}{\partial p_A}$ <0 and $\frac{\partial \Pi_{A2}}{\partial p_A}$ >0, the overall profit function has two peaks, one at p^*_{A1} and the other at p^*_{A2}, where $p^*_{A1} < p^*_{A2}$. We need to compare $\Pi_{A1}(p^*_{A1}; z)$ and $\Pi_{A2}(p^*_{A2}; p_B)$ to ascertain firm A's globally optimal price.

From the definition of $\overline{p}_B$, we have $\Pi_{A1}(p^*_{A1}) \geq \Pi_{A1}(\frac{z}{q-z}\overline{p}_B) = \Pi_{A2}(\frac{z}{q-z}\overline{p}_B) = \Pi_{A2}(p^*_{A2})$. Thus, firm A's global optimization occurs at p^*_{A1}. From the definition of $\underline{p}_B$, we have $\Pi_{A2}(p^*_{A2}) \geq \Pi_{A2}(\frac{z}{q-z}\underline{p}_B) = \Pi_{A1}(\frac{z}{q-z}\underline{p}_B) = \Pi_{A1}(p^*_{A1})$. Thus, firm A's global optimization occurs at p^*_{A2}. Since $\Pi_{A2}(p^*_{A2}; p_B)$ is continuously decreasing in p_B while $\Pi_{A1}(p^*_{A1})$ is independent of p_B, there exists a unique $\tilde{p}_B$ strictly between $\underline{p}_B$ and $\overline{p}_B$ such that $\Pi_{A1}(p^*_{A1}) = \Pi_{A2}(p^*_{A2}; \tilde{p}_B)$.

Combining the above cases yields Lemma 4. Q.E.D.

Proof of Proposition 8

The firms' best response curves are depicted as follows as they help to develop the proof.

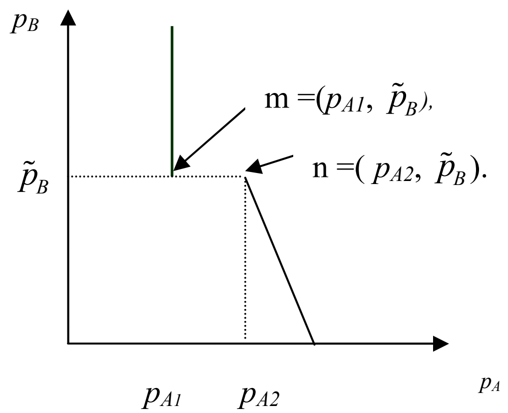


Figure (a). Firm A's best response

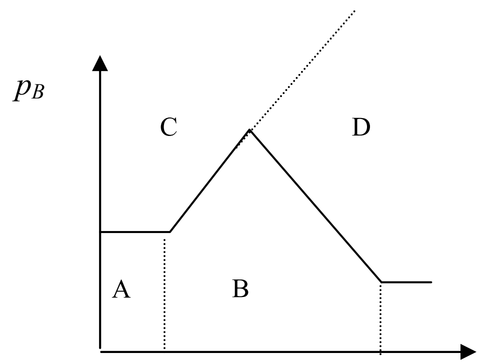


Figure (b). Firm B's best response

When we combine the two best response curves, the existence and characteristics of the Nash equilibrium depend on where the two points, m and n, in Figure (a) are located in figure (b). In the uniform distribution case, we can show the following possibilities:

- If $z > z_{max}$, then m is located in area A, so the situation is as depicted in Figure 8 (a);
- If $z_{min} < z < z_{max}$, m is located in area C and n is located in area B. Since $p_A = \frac{q - p_B + c_A}{2}$ has a steeper slope than $p_B = \frac{q - p_A + c_B}{2}$, the situation is as depicted in Figure 8 (b);
- If $z < z_{min}$, m is located in area C and n is located in area D, so the situation is as depicted in Figure 8 (c).

If $z_{min} < z < z_{max}$, then there is no pure-strategy Nash equilibrium, but a mixed-strategy Nash equilibrium can be constructed: firm B sets its price at $\tilde{p}_B$ whereas firm A sets its price at p_{A1} with probability α and at p_{A2} with probability $(1-\alpha)$.

Let us analyse firm B's best response to the mixed strategy. Let y_1 and y_2 denote firm B's best response to p_{A1} and p_{A2}, respectively. Since m is in area C and n is area B, we have $y_1 < \tilde{p}_B < y_2$. Regardless of whether firm A sets its price p_{A1} or p_{A2}, we can show that in terms of firm B's profit any p_B higher than y_2 is dominated by y_2 and that any p_B lower than y_1 is dominated by y_1. Thus, firm B's optimal price to firm A's randomization is between y_1 and y_2.

When firm B sets its price p_B strictly in between y_1 and y_2, we have $\frac{q-z}{z}\ p_{A1} < p_B < \frac{q-z}{z}\ p_{A2}$, implying that firm B's profit function is given by

$$\Pi_B(p_B) = \alpha\,(1-\frac{p_B}{q-z})(p_B-c_B) + (1-\alpha)(1-\frac{p_A+p_B}{q})\,(p_B-c_B)$$

If α=0, firm B's optimal price is y_2; if α=1, firm B's optimal price is y_1. Since $\tilde{p}_B$ is between y_1 and y_2 and firm B's optimal price is continuous in α, by the mean value theorem there exists α such that firm B's optimal price is equal to $\tilde{p}_B$. This proves that $[\alpha, p_{A1}, p_{A2}, \tilde{p}_B]$ constitutes a mixed-strategy Nash Equilibrium. Q.E.D.

Figures

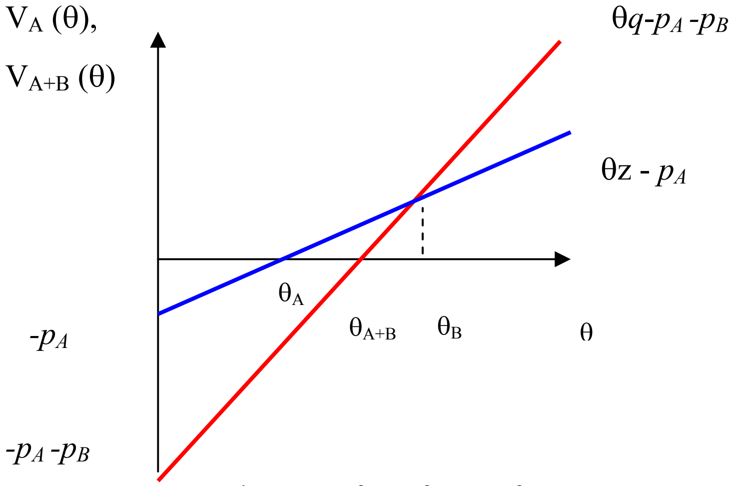


Figure 1. $\theta_A < \theta_{A+B} < \theta_B$
Virtually Independent Products

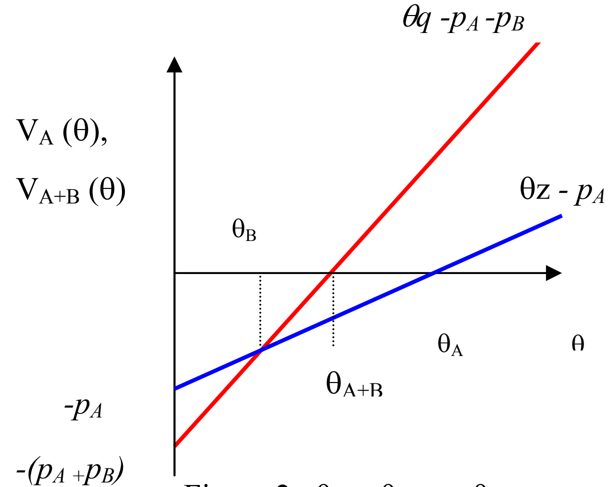


Figure 2. $\theta_B < \theta_{A+B} < \theta_A$
Virtually Strict Complements

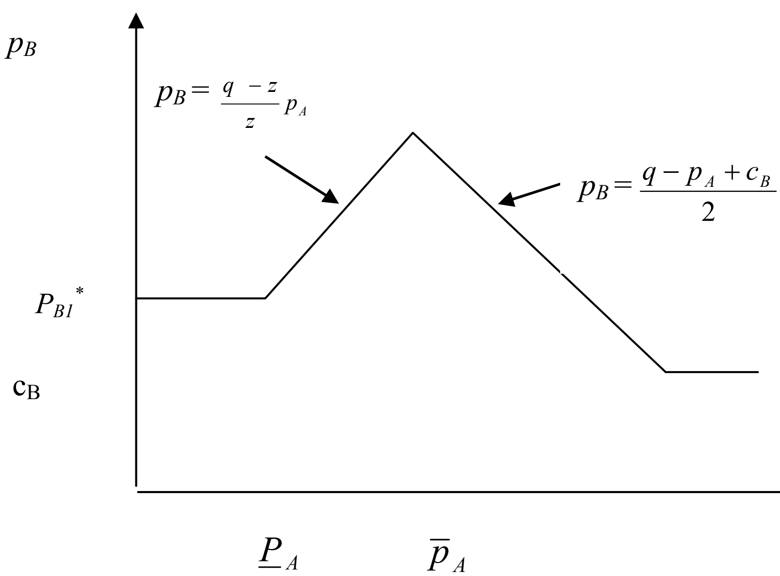


Figure 3. Firm B's best response to p_A if θ is uniformly distributed

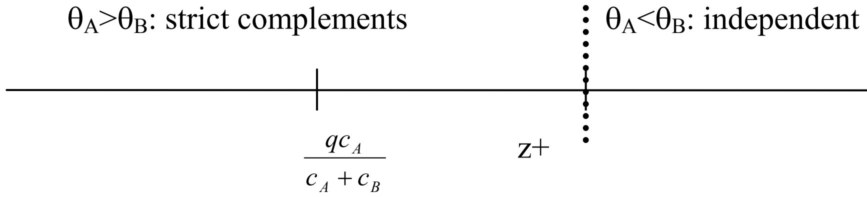
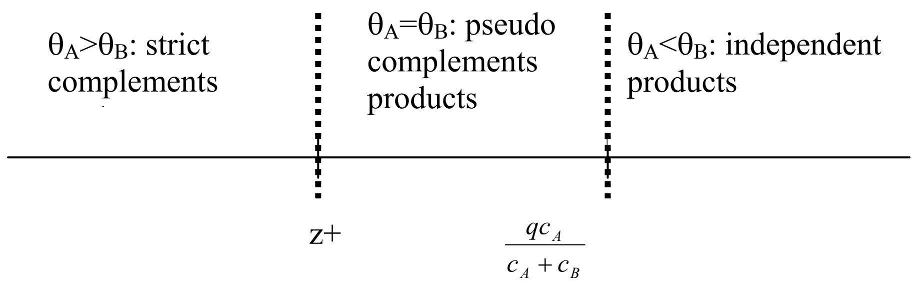


Figure 4. Product relationships between A and B



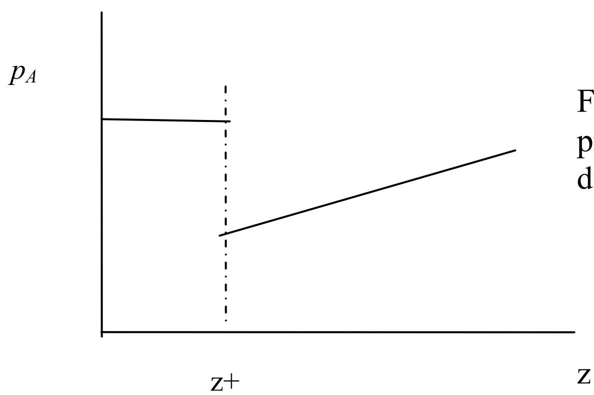


Figure 5. Firm A's optimal price if θ is uniformly distributed

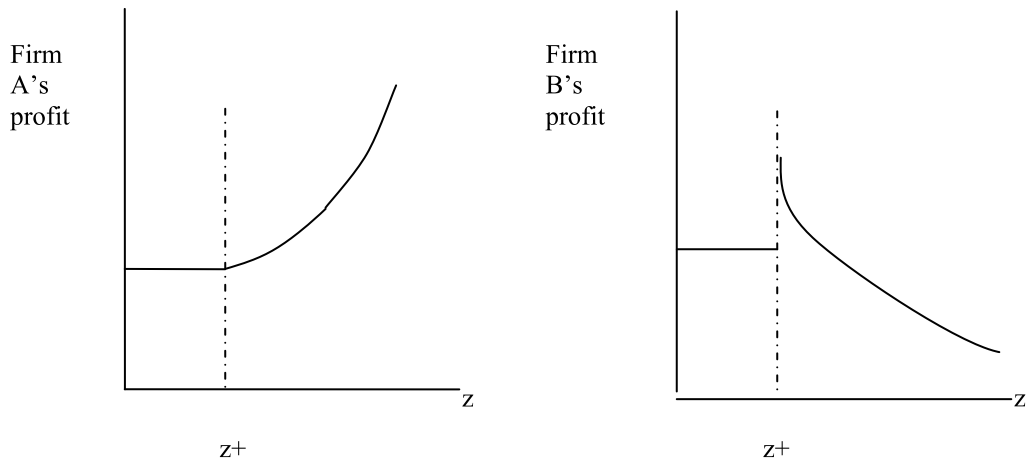


Figure 6. Firm A and B's profits if θ is uniformly distributed

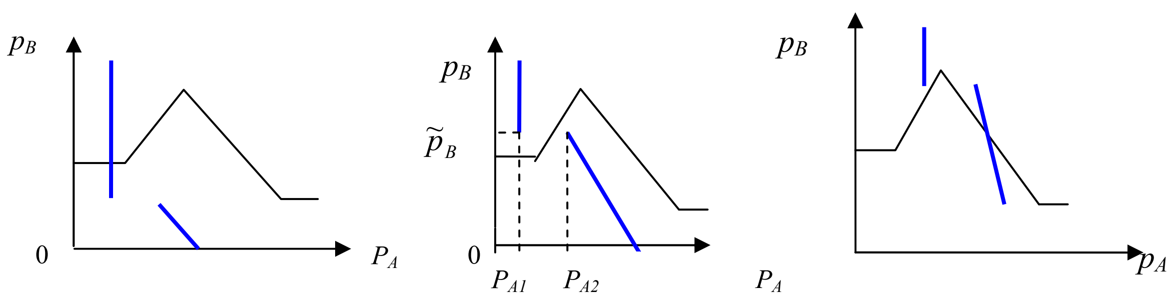


Figure 7-(a), $z > z_{max}$ Figure 7-(b), $z_{min} < z < z_{max}$ Figure 7-(c), $z < z_{min}$

Figure 7-(a), (b), (c): The existence of a pure-strategy Nash equilibrium if θ is uniformly distributed.

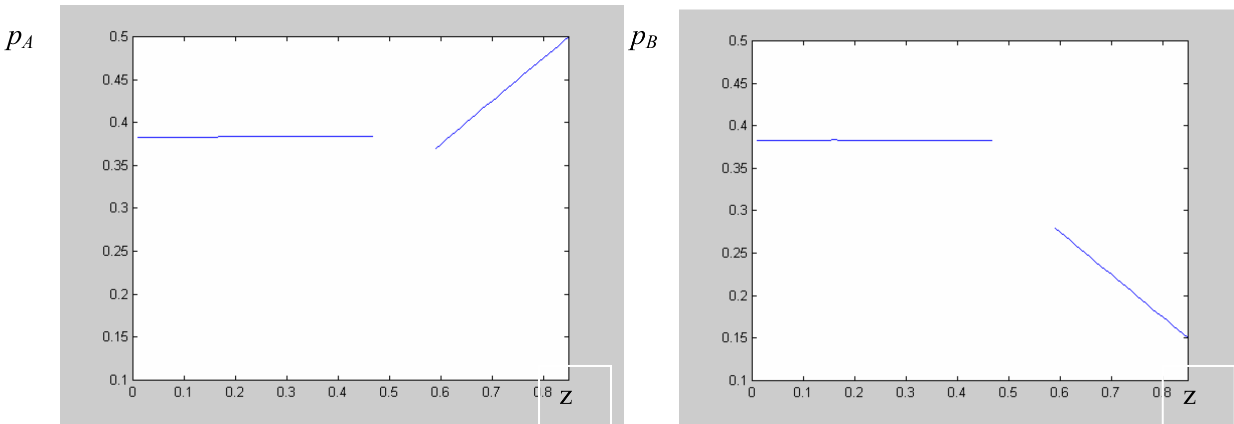


Figure 8. p_A and p_B in a pure-strategy Nash Equilibrium for the case q=1, $c_A = c_B$ = 0.15 when firms A and B set prices simultaneously

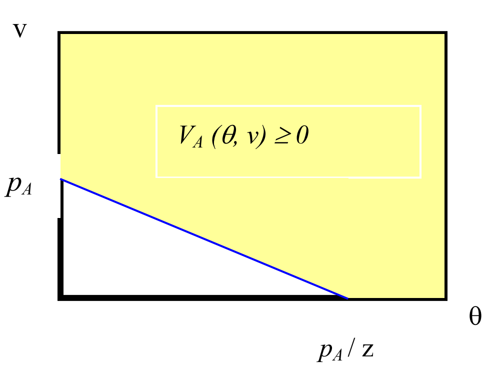


Figure 9 (a)

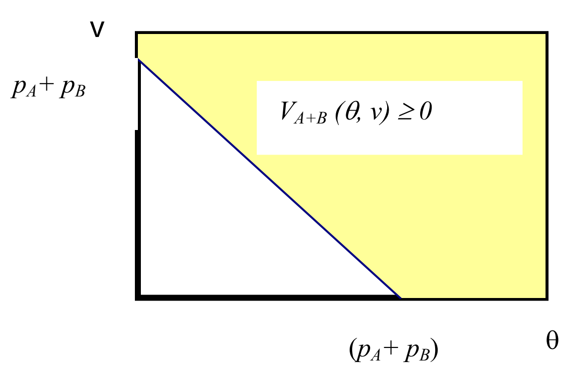


Figure 9 (b)

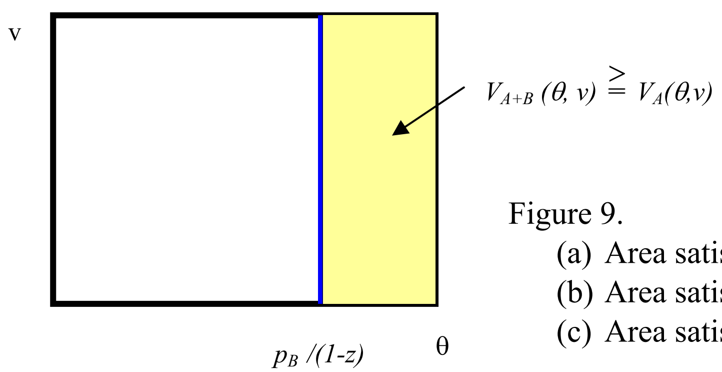


Figure 9.
(a) Area satisfying $V_A(\theta, v) \geq 0$
(b) Area satisfying $V_{A+B}(\theta, v) \geq 0$
(c) Area satisfying $V_{A+B}(\theta, v) \geq V_A(\theta, v)$

Figure 9 (c)

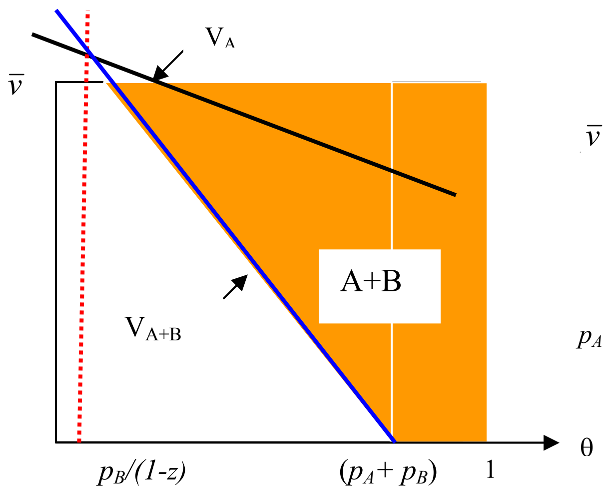


Figure 10 (a) Strict complements

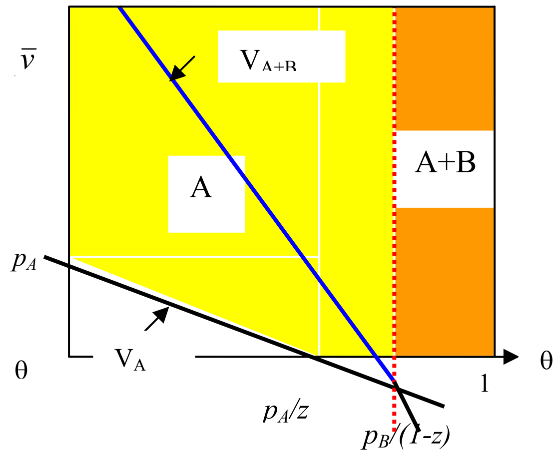


Figure 10 (b) Independent products

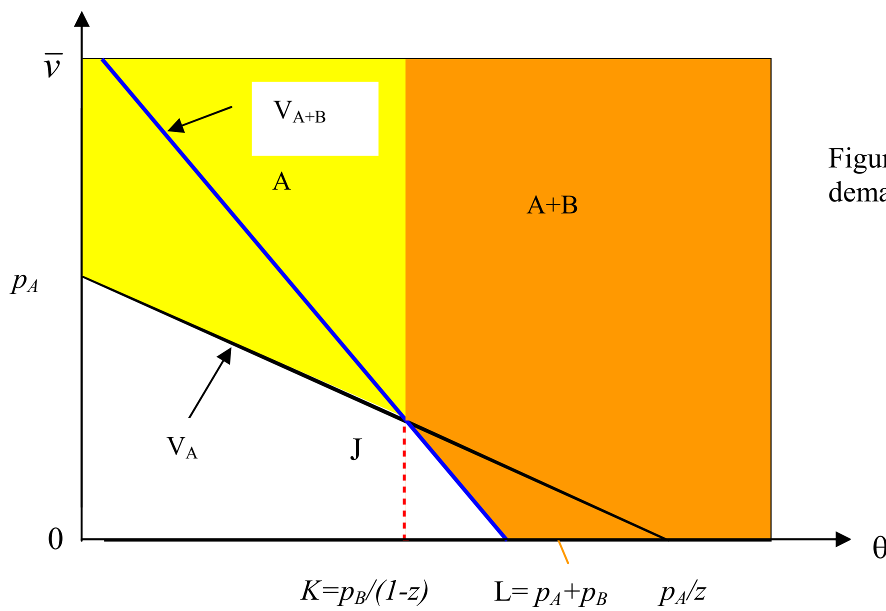


Figure 10 (c). Mixed demand

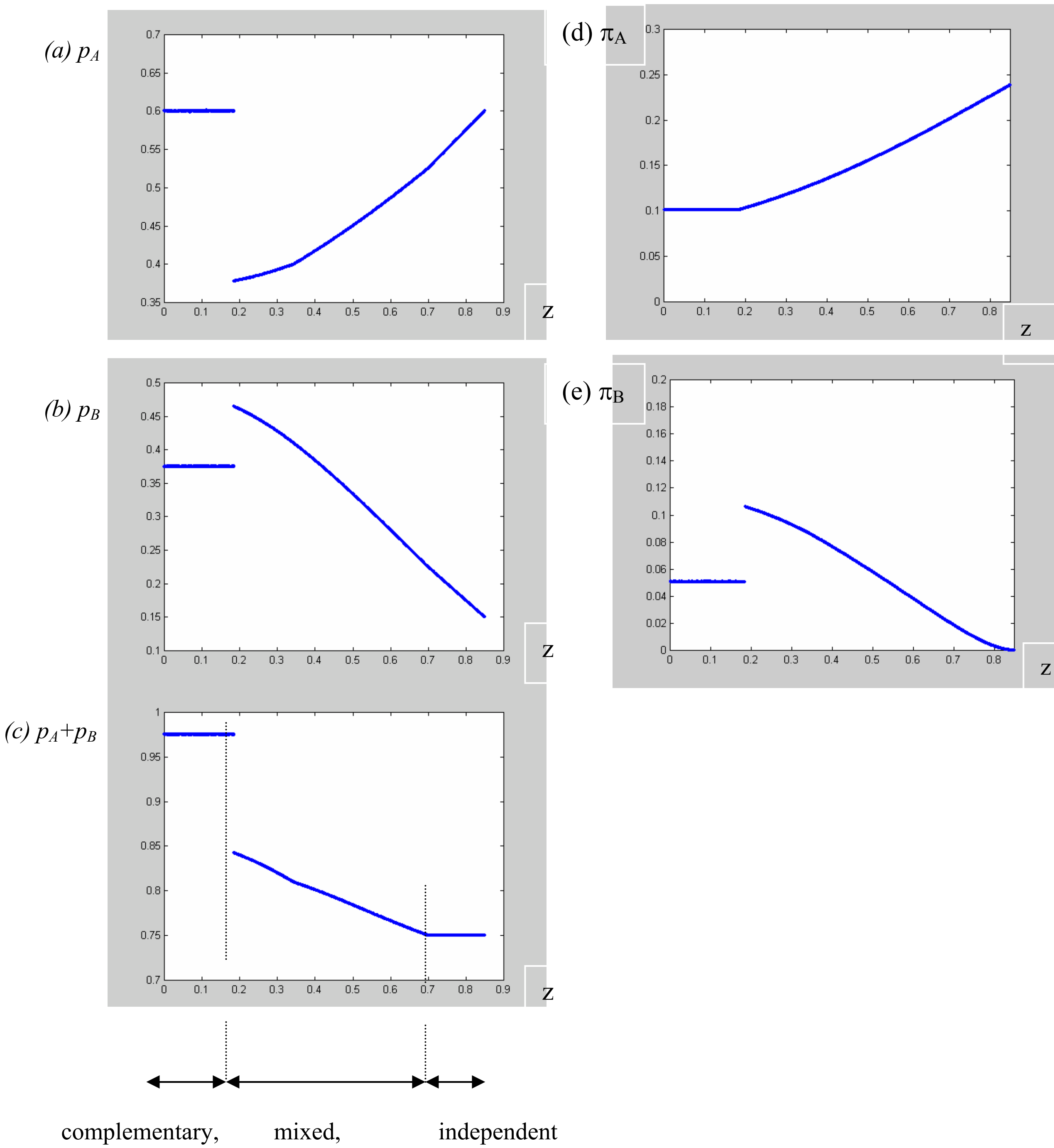


Figure 11. The firms' optimal prices in the numerical simulation when $c_A = c_B = 0.15$ and $\bar{v} = 0.4$
(a) The effect of z on p_A, (b) The effect of z on p_B, (c) The effect of z on $(p_A + p_B)$
(d) The effect of z on π_A, (e) The effect of z on π_B.

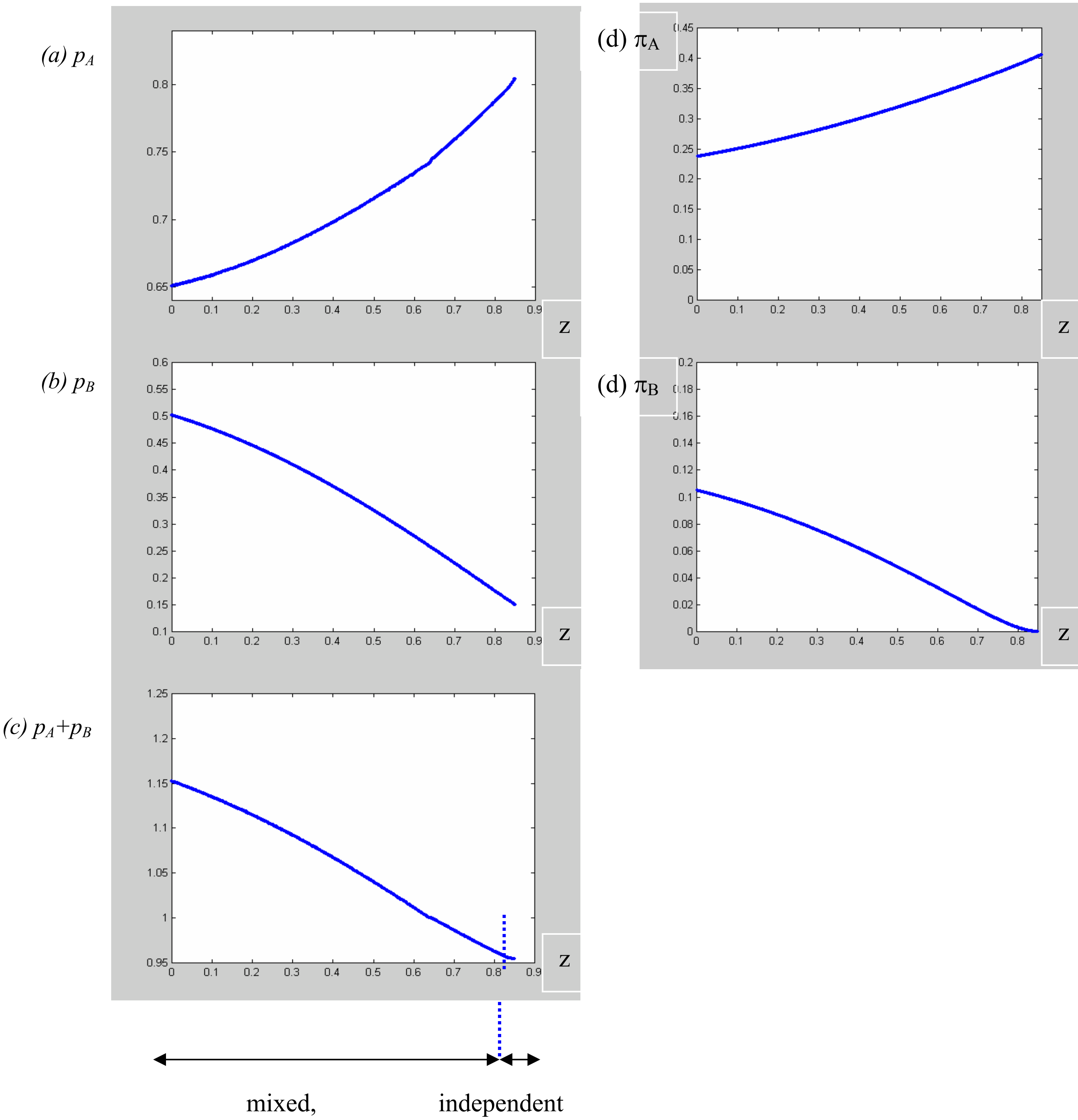


Figure 12. The firms' optimal prices in the numerical simulation when $c_A = c_B = 0.15$ and $\bar{v} = 1$
(a) The effect of z on p_A, (b) The effect of z on p_B, (c) The effect of z on $p_A + p_B$, (d) The effect of z on π_A, and (e) The effect of z on π_B.

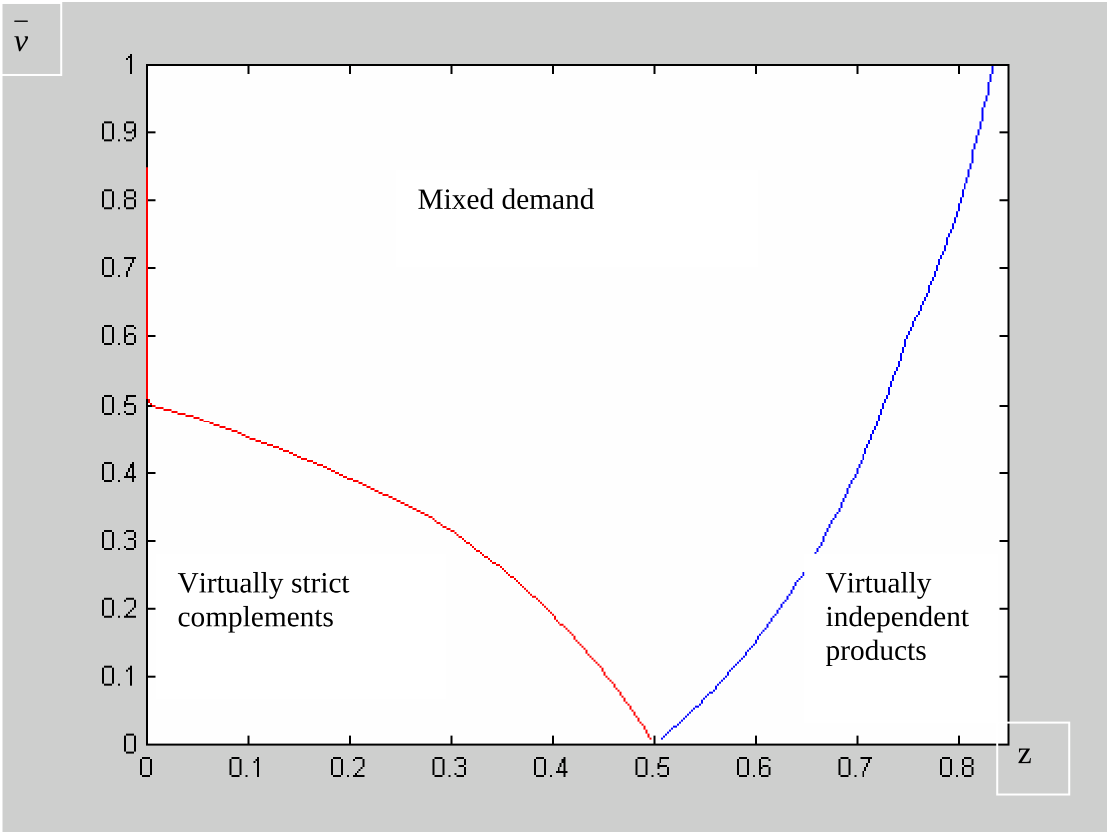


Figure 13. This figure shows how the demand regime depends on z and $\bar{v}$ when $c_A = c_B = 0.15$.