Inflation and Growth: Impatience and a Qualitative Equivalence[*]

Been-Lon Chen[**]

Institute of Economics, Academia Sinica

Mei Hsu

Department of Economics, National Taipei University

Chia-Hui Lu

Department of Economics, National Taiwan University

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Abstract

This paper studies the role of an endogenous time preference on the relationship between inflation and growth in the long run in both the money-in-utility-function (MIUF) and transaction costs (TC) models. We establish a qualitative equivalence between the two models in a setup without a labor-leisure tradeoff. When the time preference is decreasing (or increasing) in consumption and real balances, both the MIUF and TC models are qualitatively equivalent in terms of predicting a negative (or positive) relationship between inflation and growth in a steady state. Both a decreasing and an increasing time preference in consumption are consistent with the arguments in the literature. While a decreasing time preference in real balances corroborates with empirical evidence, there is no evidence in support of an increasing time preference in real balances.

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[**] Corresponding address: Been-Lon Chen, the Institute of Economics, Academia Sinica, 128 Academia Rd., Section 2, Taipei 11529, TAIWAN. Phone: (886-2)27822791 ext. 309; Fax: (886-2)27853946; Email: bchen@econ.sinica.edu.tw

1. Introduction

The relation between inflation and growth has been one of the central issues in macroeconomic literature since the work by Tobin (1965) and Sidrauski (1967a, b). It is well-known that alternative approaches of introducing money yield incompatible predictions concerning the effect of anticipated inflation on capital accumulation. These competing approaches include the money-in-utility-function (MIUF) model, the transactions-costs (TC) approach and the cash-in-advance (CIA) approach.[1] A comparison of the MIUF and TC approaches was given by Dornbush and Frenkel (1973), and the equivalence of the MIUF and TC approaches was first indicated by an example in Brock (1974).[2] Feenstra (1986) established *functional* equivalence between the MIUF and TC approaches.[3] Since then, numerous theoretical developments in comparisons of these competing approaches have been achieved.

Among these, there are two successful advancements that established a *qualitative* equivalence between alternative approaches. The first is Wang and Yip (1992). Different from Feenstra, Wang and Yip (1992) used the utility employed in Brock (1974) that allowed for direct utility interactions between leisure and real balances. A qualitative equivalence was created when higher monetary growth leads to lower capital and higher leisure in a steady state. The required conditions were weaker than those necessary for a functional equivalence. Next, Zhang (2000) also obtained a *qualitative* equivalence between inflation and growth, but unlike Wang and Yip (1992), the equivalence was among different cases in the TC approach. With labor-leisure tradeoffs, Zhang (2000) considered a general TC function and assumed that money holdings reduce the transaction costs of consumption, production or investment. He obtained comparative-static results similar to Wang and Yip (1992) when assuming a dominant consumption effect.

These existing studies represented preferences with labor-leisure tradeoffs and a functional in which a utility function is discounted by a constant rate of time preference. The specification is attractive because it is analytically tractable and easily describes how tastes and opportunities interact to determine an economy's paths of consumption and capital accumulation. However, its rigid structure severely limits the conclusions and explanatory power of the corresponding models.

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[1] While Tobin (1965) used the TC approach in a non-optimized model and obtained a positive relationship between inflation and growth as a result of portfolio shift from non-interest bearing real balances to capital, Sidrauski (1967a, b) employed the MIUF approach with individuals' rational choices in an infinite horizon model and found growth independent of money growth or inflation, known as the super-neutrality of money.

[2] Dornbush and Frenkel (1973) used a non-optimized model with inelastic labor, whereas Brock (1974) employed an individual optimized, infinite-horizon model with leisure and thus, elastic labor.

[3] In a model without capital and labor-leisure choice, Feenstra showed a duality between the two approaches by redefining choice variables, but he required unpopular utility specifications.

Their preference implies the marginal rate of substitution between time t_1 and t_2 is independent of consumption at any time $t \neq t_1$, t_2. As a result, when a labor-leisure tradeoff is not possible, this kind of model implies the neutrality of money in the long run. This paper considers a generalized class of preferences which has an attractive feature in which the rate of time preference is endogenous. This class of preferences allows the demands of consumption and real balances at any time $t \geq t_1$ to affect the marginal rate of substitution between time t_1 and $t_2 > t_1$. Thus, if there is a shock affecting consumption and real balances now, the marginal rate of substitution between now and future is influenced. As a result, even if a labor-leisure tradeoff is not possible, capital accumulation and growth may be changed. In this paper, we study the economic implications on the relationship between inflation and growth in the long run when a class of preference with an endogenous time preference is considered.

Specifically, in the present model, we study the non superneutrality of money in an otherwise standard optimal growth model. The departure here is to take account of an endogenous time preference. Labor supply is perfectly inelastic as is usually assumed in optimal growth models. We choose the setup of inelastic labor not because it is more realistic. Rather, existing studies by Brock (1974) and Wang and Yip (1992) have relied on elastic labor in order to establish non superneutrality.[4] Our setup highlights the significance of an endogenous time preference in establishing the non superneutrality result without relying on leisure-labor tradeoffs.

An endogenous time preference has been stressed at least as early as Böhm-Bawerk (1989). Fisher (1930, pp61-94) has observed the changes in the rate of time preference, or the degree of impatience, over time as consumption, income, risks and personal factors change. Koopmans (1960) has elaborated an endogenous time preference in a model class with a recursive utility. In a neoclassical growth framework, endogenous impatience was first formalized by Uzawa (1968), followed by Wan (1970). Endogenous impatience has since then been extensively used in optimal growth models (e.g., Lucas and Stokey, 1984; Epstein, 1987a; Obstfeld, 1990), endogenous growth models (e.g., Palivos et al, 1997) and growth models with open economies (e.g., Devereux and Shi, 1991).[5]

In a standard optimal growth model, steady-state capital is determined by the commodity market clearance condition and the Keynes-Ramsey rule. In both the MIUF and TC models a higher rate of monetary expansion reduces real balances because of higher inflation and thus, a

[4] Another way to obtain non-neutrality in the long run independent of labor-leisure tradeoffs is to introduce to wealth in utility as performed by Gong and Zuo (2001) and Chang and Tsai (2003), among others.

[5] Recently, Becker and Mulligan (1997) proposed the hypothesis of endogenously reducing subjective discount on the future utilities as a result of consumers' efforts.

higher opportunity cost of holding money. In our study, the effect of anticipated inflation on real balances is transmitted to capital via an endogenous degree of impatience. It is easy for money to become non superneutral in a steady state because of an endogenous response of a time preference. We establish a qualitative equivalence between the MIUF and TC approaches, under not only a positive but also a negative relationship between inflation and growth. The positive relationship, a variation on Tobin effect, emerges when the time preference is *increasing* in both consumption and money at the same time, while the negative relationship, a reverse Tobin effect, arises when the time preference is *decreasing* in both consumption and money at the same time. As a result, the long-run relationship between inflation and growth depend crucially on the response of a time preference to consumption and money. There is evidence consistent with a time preference both decreasing and increasing in consumption. However, there is only evidence in support of a time preference decreasing in real balances, as was partly evidenced by Becker and Mulligan (1994). Under plausible decreasing impatience, we thus establish a qualitative equivalence between the MIUF and TC approaches in line with that in Wang and Yip (1992) and Zhang (2000). Different from these two existing studies, our equivalence result relies on neither a labor-leisure tradeoff nor a dominant consumption effect.

Finally, we must mention that this paper is not the first attempt to analyze the implication of an endogenous time preference on the non superneutrality. Earlier, Epstein and Hynes (1983, Sec. V) have argued because of the substitutability between real balances and consumption in a MIUF model, inflation increases steady-state capital when the time preference is endogenous. Thus, only the Tobin effect is observed in Epstein and Hynes. Our model renders the result in Epstein and Hynes (1983) as a special case only when real balances increase impatience. We predict a negative relationship between inflation and capital when real balances decrease impatience. Moreover, we establish a qualitative equivalence between the MIUF and TC models, an unaddressed issue in Epstein and Hynes (1983).

The remainder of this paper is organized as follows. Section 2 sets up a MIUF model and studies the optimization conditions. Section 3 examines money superneutrality in the MIUF economy. Section 4 studies the TC model and establishes a qualitative equivalence between the MIUF and TC models. Finally, some concluding remarks are made in Section 5.

2. A MIUF Model

We consider an extension of the one-sector optimal growth model with infinitely lived agents. The economy consists of a large number of identical agents, normalized to unity, and each of them has one unit of labor which is supplied inelastically. The agent seeks to maximize the following discounted sum of lifetime felicities

$$U = \int_0^\infty u(c(t), m(t)) X(t) dt \,, \tag{1}$$

in which u is the felicity function, c and m are individual consumption and real balances, and X is the cumulative discount at time t. Following Sidrauski (1967a, b), money directly enters the felicity, on the argument that this represents a reduced form equation in a world of transaction costs. By facilitating transactions, money yields a direct utility to the agent that is not associated with other assets such as capital, which then further yields an indirect utility through the income they generate and the consumption goods they enable the agent to purchase.

The cumulative discount is endogenous, and is $X(t) \equiv \exp[-\int_0^t \rho(c(\tau), m(\tau)) d\tau]$, where $\rho(\cdot, \cdot)$ is the discounting function, or the degree of impatience. This relation is equivalent to

$$\dot{X} = -\rho(c(t), m(t)) X(t) \text{, with } X(0) \text{ given,} \tag{2}$$

Assumption 1. *(i)* $u_c(c,m) > 0 > u_{cc}(c,m)$, $u_m(c,m) > 0 > u_{mm}(c,m)$, *for any c>0 and m>0***;**

(ii) $\frac{u_{cc}}{u} - \frac{\rho_{cc}}{\rho} < 0$, *for any c>0 and m>0.*

The assumption in *(i)* postulates a positive and decreasing marginal utility of consumption; the same assumption is made for real balances following existing studies of MIUF models. The assumption in *(ii)* requires the curvature of felicity with respect to consumption larger than that of discount rate with respect to consumption. This assumption is necessary in order to assure a positive intertemporal elasticity of substitution. Unlike Asako (1983) who assumed the utility is separable and Hayakawa (1995) who assumed the perfect complementarity of consumption and real balances so $u_{cm}(c, m)=u_{mc}(c, m)=0$, we allow for the substitutability between consumption and real balances without imposing restrictions on $u_{cm}(c, m)$ and $u_{mc}(c, m)$.

A correlation between consumption and time preferences is well postulated. There is, however, a considerable disagreement over whether impatience should increase or decrease as actual consumption rises. Fisher (1930) proposed that a person's impatience decreases as the economy develops. Koopmans (1960) made arguments in favor of decreasing impatience. Authors like Blanchard and Fischer (1989) and Barro and Sala-i-Martin (1995) find counterintuitive that people would be more impatient as their level of consumption rises. On the other hand, Epstein (1987a, pp73-74), who surveys the debates on this issue, offers several counter-arguments and argues that the proper interpretation of a discount rate is that individuals who know that will have a large level of consumption in the future evaluate current consumption more highly. Lucas and Stokey (1984) point out that increasing impatience, a type of diminishing returns to savings, is often needed to produce unique, stable and non-degenerate steady-state wealth distributions in a deterministic infinite-horizon setting with a fixed set of agents. Authors like Obstfeld (1990) and Palivos, et al. (1997) follow the assumption of increasing impatience in order to assure the stability. In this paper, we allow for the possibilities of both $\rho_c>0$ and $\rho_c<0$. Empirical evidence is also mixed in support of either one of the two relationships.[6]

The correlation between real cash holdings and time preference is justified as follows. In the formulation with consumption and money in utility, the rate of time preference at time t is the Volterra derivative of the present value (in utility terms) due to an upward perturbation of the consumption path at time equal and larger than t, according to Epstein and Hynes (1983) and Epstein (1987b). Applying this method, Obstfeld (1990) showed that the rate of time preference depends on consumption and the shadow price, or the co-state variable, of the cumulated discount. The shadow price of the cumulated discount at time t represents the discounted present value of the future felicities at and after time t. When both consumption and money appear in the felicity function, both arguments are in relation to the rate of time preference through the shadow price of the cumulated discount. As a result, money is in relation to time preferences. While there is little evidence about a relation between money and time preferences, Becker and Mulligan (1994) have used data in the Panel Study of Income Dynamics and uncovered a positive relationship between wealth and patience.[7] As money accounts for a fraction of wealth, their result may be viewed as supportive of the negative relationship between money and time preferences, namely $\rho_m<0$. Although such a relation is what will be focused on, in the analysis that follows we do not rule out the possibility of a positive relationship between money and impatience, exemplified by $\rho_m>0$.

The agent owns the shares of firms. The representative firm is endowed with a technology $y=f(k)$, where y is per-capita output and k is per-capita capital stock, with $k(0)$ given initially, and, for simplicity, is assumed not to depreciate.

Assumption 2. *(i)* $f_k(k)>0>f_{kk}(k), f(0)=0, \lim_{k\to\infty} f_k(k)=0$, and $\lim_{k\to 0} f_k(k)=\infty$;

(ii) $\frac{f_{kk}}{f_k}<\rho_c$, *for all k and c.*

[6] Evidence on the relationship between time preferences and consumption is based on either cross-sectional or time-series data. Hong (1988) proposed trade is likely to decrease the rate of time preference of a developing country. He uses a cross section of 42 developing countries and found evidence that supports his argument. Using household cross-section data based upon the Panel Study of Income Dynamics, Lawrence (1991) uncovered the time preference rate of the poor is three to five percentage points higher than those of the rich. Using the post-war annual time-series data from Japan and Taiwan, Ogawa (1993) found the time preference rates were declining up to a certain point and then increasing as the two economies grew. See also evidence cited in Becker and Mulligan (1997) concerning the hypothesis that (a) patience varies across individuals and (b) wealth causes patience.

[7] See also Becker and Mulligan (1997) that contains an extensive overview of the empirical studies on the connection between wealth and time preferences.

While the assumption of a positive and concave technology with the Inada conditions in *(i)* is standard in a neoclassical production function, *(ii)* is a technical condition that requires the technology be more concave than the instantaneous discount. This latter condition is necessary in order to satisfy the Correspondence Principle (Samuelson, 1948).[8]

Nominal money supply is assumed to grow at a constant rate μ with the amount of nominal money supply given initially. The real transfer from the government, v, is financed by seigniorage, so $v=\mu m$. Unspent real disposable income accumulates wealth. The budget constraint of the representative agent is thus

$$\dot{a} = f(k) - \pi m + v - c, \tag{3}$$

where $a=k+m$ is the agent's total wealth and π is the rate of inflation.

The representative agent's optimal program is to maximize (1), subject to (2) and (3) and the wealth constraint. The necessary conditions are

$$\theta = u_c(c,m) - \lambda\rho_c(c,m), \tag{4a}$$

$$\theta f_k(k) = \xi, \tag{4b}$$

$$u_m(c,m) - \lambda\rho_m(c,m) = \theta\pi + \xi, \tag{4c}$$

$$\dot{\lambda} = -u(c,m) + \lambda\rho(c,m), \tag{4d}$$

$$\dot{\theta} = \rho(c,m)\theta - \xi, \tag{4e}$$

along with the transversality conditions $\lim_{t\to\infty}\lambda(t)X(t) = 0$ and $\lim_{t\to\infty}\theta(t)X(t)a(t) = 0$, where $\theta>0$ which is the co-state variable associated with total wealth, $-\lambda>0$ is the co-state variable associated with the impatience, and $\xi>0$ is the Lagrange multiplier associated with wealth.

While (4a) equates the marginal cost to the marginal utility of consumption, (4b) and (4c) require the marginal product of capital and the discounted marginal utility of real balances equal the shadow price of capital and real balances. Finally, (4d) and (4e) are the Euler equations governing the changes in the shadow price of the cumulative discount and capital. Note that compared to the standard optimal one-sector growth model, there are two differences here. The first difference is in (4a) and (4c) where time preferences change with respect to consumption and real balances and affect the marginal utility of consumption and real balances. Moreover, as the *individual* cumulative discount changes, its shadow price changes over time in (4d), in contrast to a set up without the shadow price of a *social* cumulative discount (e.g., Meng, 2006).

In equilibrium, both the money and the goods markets must be clear. That is,

[8] The condition is thus a variant of the Brock-Gale condition that requires the increase in the discount rate to dominate the increase in the marginal product of capital in the steady-state equilibrium.

$$\dot{m} = (\mu - \pi)m, \tag{5a}$$

$$\dot{k} = f(k) - c. \tag{5b}$$

A Perfect Foresight Equilibrium (PFE) is a time path $\{c, m, k, \lambda, \theta, \xi, \pi\}$ that satisfies optimization conditions (4a)-(4e) and the money and commodity market equilibrium, (5a) and (5b). We may derive the dynamic equilibrium system as follows. Using (4a)-(4c), the money market equilibrium condition is rewritten as

$$\dot{m} = \left(\mu - \tfrac{u_m(c,m) - \lambda\rho_m(c,m)}{u_c(c,m) - \lambda\rho_c(c,m)} + f_k(k)\right)m. \tag{6a}$$

Moreover, total differentiation of (4a) yields

$$(u_{cc} - \lambda\rho_{cc})\dot{c} + (u_{cm} - \lambda\rho_{cm})\dot{m} - \rho_c\dot{\lambda} = \dot{\theta},$$

and substituting into (4d)-(4e) and (6a) along with the use of (4a)-(4c) leads to

$$\dot{c} = \tfrac{-1}{u_{cc} - \lambda\rho_{cc}}[(f_k - \rho)(u_c - \lambda\rho_c) - \rho_c(\lambda\rho - u) + (u_{cm} - \lambda\rho_{cm})\left(\mu - \tfrac{u_m - \lambda\rho_m}{u_c - \lambda\rho_c} + f_k\right)m], \tag{6b}$$

in which $\frac{-1}{u_{cc} - \lambda\rho_{cc}}$ is the intertemporal elasticity of substitution and Assumption 1.2 assures the intertemporal elasticity is positive in a steady state where $\lambda = u/\rho$.

Finally, both the Euler equation for the cumulative discount and the goods market clearance are in (4d) and (5b). Thus, the system is reduced to four dynamic equations, (4d), (5b), (6a) and (6b). These four equations determine the time path of c, m, k and λ. The time path of the shadow prices θ, ξ and π are in turn determined by (4a)-(4c).

3. A Steady State in a MIUF Economy

A steady state is a PFE when $\dot{c} = \dot{m} = \dot{k} = \dot{\lambda} = \dot{\theta} = 0$.[9] In a steady state, while (5a) indicates $\pi^* = \mu$, we may simplify the conditions in (4b), (4d)-(4e), (5b) and (6a)-(6b) as follows.[10]

$$f_k(k^*) = \rho(c^*, m^*), \tag{7a}$$

$$f(k^*) = c^*, \tag{7b}$$

$$\lambda^* = \tfrac{u(c^*, m^*)}{\rho(c^*, m^*)}, \tag{7c}$$

$$f_k(k^*) + \mu = \tfrac{u_m(c^*, m^*)/u(c^*, m^*) - \rho_m(c^*, m^*)/\rho(c^*, m^*)}{u_c(c^*, m^*)/u(c^*, m^*) - \rho_c(c^*, m^*)/\rho(c^*, m^*)}, \tag{7d}$$

$$\xi^* = \rho(c^*, m^*)[u_c(c^*, m^*) - \lambda^*\rho_c(c^*, m^*)]. \tag{7e}$$

Equations (7a), (7b) and (7d) determine the values for $\{c^*, k^*, m^*\}$ in a steady state. Then,

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[9] An asterisk is used to denote a steady-state value.

[10] While these equations are easily obtained, in deriving (7d) equations (4a)-(4c), (5a) and (7c) are used.

we use (7c) to determine λ^* and, finally, (7e) to determine ξ^*.

We are ready to analyze the relationship between money and growth in the long run. This involves a comparative-static exercise around a steady state about the effect of a higher growth rate of monetary expansion, i.e., a higher μ. A meaningful comparative-static exercise requires that the steady state be a stable saddle. In the Appendix, we take a linear Taylor expansion of system (4d), (5b), (6a) and (6b) around steady-state $\{c^*, k^*, m^*, \lambda^*\}$. A negative determinant of the Jacobean matrix of the linear system is required in order to guarantee saddle stability. Denote by Δ the determinant of the Jacobean matrix. Then, the condition corresponds to $\Delta<0$. Under the saddle stability condition, the equilibrium time path around the steady state is locally determinate.

To investigate the relationship between money and growth in the long run, we start with the special case when real balances do not affect the degree of impatience, followed by the general case when real balances affect the degree of impatience.

Consider the special case when real balances do not affect the degree of impatience, $\rho_m=0$. Then, the steady-state Keynes-Ramsey rule (7a), for convenience referred to as the KR rule, and the steady-state commodity market clearance condition (7b), referred as the CC condition, simultaneously determine the unique level of capital and consumption. Specifically, while the steady-state CC condition is positively sloping in the (k, c) plan, the steady-state KR rule may be negative or positive sloping depending on $\rho_c\geq0$ and $\rho_c\leq0$ (Figures 1 and 2). Under $\rho_c\leq0$, the steady-state KR rule must be steeper than the CC condition in the (k, c) plan in order to satisfy the Samuelson Correspondence Principle.[11] The relative slopes of the two loci imply the requirement of $f_{kk}-\rho_c\rho<0$ in Assumption 2.2. Thus, under $f_{kk}-\rho_c\rho<0$, there exists a unique steady state.

[Insert Figures 1 and 2]

In both conditions of $\rho_c\geq0$ and $\rho_c\leq0$, the steady-state levels of consumption and capital are completely determined by these two conditions in (7a) and (7b) independent real balances (E_0 in Figures 1 and 2). Then, the growth rate of nominal money, μ, exerts effects on real balances completely determined by (7d). In this economy, nominal money is thus superneutral in the long run even though real balances interact in a utility term with consumption and consumption affects time preferences. Indeed, as we have shown in the Appendix, this special superneutrality feature is shared with a cash-in-advance constraint on consumption.

Next, consider the general case when real balances affect the degree of impatience, i.e., $\rho_m\neq0$. In this case, it may be $\rho_m>0$ and $\rho_m<0$. The growth rate of money affects real balances, and

[11] Under $\rho_c\leq0$, should the steady-state KR rule be flatter than the CC condition, then a lower productivity shock would have led to an increase, rather than a decrease, in both capital and consumption in steady state.

via the effect on impatience affects consumption and capital.

To specifically understand the effects, first, if we differentiate (7a) and (7b), we obtain

$$(f_{kk} - \rho_c \rho) dk = \rho_m dm. \tag{8a}$$

where $f_{kk} - \rho_c \rho < 0$ according to Assumption 2.

Next, differentiating (7d), together (7b), yields[12]

$$\rho\{f_{kk} \tfrac{\theta}{\rho} + J_{21} \tfrac{1}{m} + \tfrac{\theta}{\rho} J_{24} \tfrac{1}{m}\} dk + \{J_{22} \tfrac{1}{m} + \tfrac{\theta}{\rho}(\rho+\mu) J_{24} \tfrac{1}{m}\} dm = -d\mu. \tag{8b}$$

where $J_{21} = -\left[\frac{u_{cm} - \lambda\rho_{cm} - (\rho+\mu)(u_{cc} - \lambda\rho_{cc})}{u_c - \lambda\rho_c}\right] m^*$, ,

$J_{22} = -\left[\frac{u_{mm} - \lambda\rho_{mm} - (\rho+\mu)(u_{cm} - \lambda\rho_{cm})}{u_c - \lambda\rho_c}\right] m^*$,

$J_{24} = -\left[\frac{-\rho_m + (\rho+\mu)\rho_c}{u_c - \lambda\rho_c}\right] m^*$.

Together (8a) and (8b), we obtain

$$\frac{dm}{d\mu} = \frac{-(f_{kk} - \rho_c \rho)}{\Lambda} < 0, \tag{9a}$$

where $\Lambda = -\frac{1}{\theta\rho m}\Delta(u_{cc} - \lambda\rho_{cc}) < 0$ as $u_{cc} - \lambda\rho_{cc} = u_{cc} - u\frac{\rho_{cc}}{\rho} < 0$ under Assumption 1 and Δ<0 when the steady state is a saddle.

It follows from (9a) that an increase in the growth rate of money unambiguously reduces the holdings of real balances. A lower level of real balances then affects the holding of capital and consumption in a steady state through (8a) and (7b).

Substituting (9a) into (8a) yields

$$\frac{dk}{d\mu} = \frac{\rho_m}{f_{kk} - \rho_c \rho} \frac{dm}{d\mu} = \frac{-\rho_m}{\Lambda} \underset{\leq}{\geq} 0 \text{ if } \rho_m \underset{\leq}{\geq} 0. \tag{9b}$$

The effect on capital thus depends on how real balances affect the degree of impatience. When the degree of impatience is increasing in real balances, capital holdings are higher in a steady state (point A in Figures 1 and 2). This is the Tobin effect, a result predicted by Epstein and Hynes (1983) in the case with an endogenous time preference. Alternatively, if the degree of impatience is decreasing in real balances, capital holdings are lower in a steady state (point B Figures 1 and 2).

Similarly, if we substitute (9b) into (7b), a higher growth rate of money affects the steady-state consumption in the same direction as that of capital,

$$\frac{dc}{d\mu} = \frac{-\rho_m f_k(k)}{\Lambda} = sign\{\frac{dk}{d\mu}\}. \tag{9c}$$

[12] See Appendix for derivation of (8b).

Intuitively, a higher growth rate of money leads to lower real balances. If the degree of impatience is increasing in real balances, an individual is more patient as he has less real balances. The representative agent tends to consume less and saves more. It follows that capital stock is higher in a steady state. As a result, output and consumption are higher in a steady state. Alternatively, if the degree of impatience is decreasing in real balances, an individual becomes less patient as real balances decrease. The representative agent then consumes more and saves less at that time. Consequently, capital stock, output and consumption are lower in a steady state.

Using (9a) and (9b), the change in wealth in the long run is

$$\frac{da}{d\mu}=\frac{dm}{d\mu}+\frac{dk}{d\mu}=\frac{-[(f_{kk}-\rho_c\rho)+\rho_m]}{\Lambda}\underset{\leq}{\geq}0 \text{ if } \rho_m \underset{\leq}{\geq} -(f_{kk}-\rho_c\rho)>0. \tag{10a}$$

If lower real balances make the agent more impatient, real balances and capital both are lower in a steady state. As a result, wealth decreases in a steady state. However, if $\rho_m>0$ and is sufficiently large, lower real balances make the agent more patient and he thus saves so much that the increase in capital is larger than the decrease in real balances. In this case, wealth is larger in a steady state.

Finally, the level of utility in a steady state is $U=\frac{u(c^*,m^*)}{\rho(c^*,m^*)}$. A lower level of real balances reduces the felicity but may decrease or increase the discount rate, so the welfare effect is ambiguous. Moreover, the level of consumption may be higher or lower, making the welfare effect even ambiguous. The net effect on the level of utility is (see Appendix)

$$\frac{dU}{d\mu}=(u_c-\lambda\rho_c)\left(\frac{\rho_m}{f_{kk}-\rho_c\rho}+1+\frac{\mu}{\rho}\right)\frac{dm}{d\mu}\underset{\leq}{\geq}0 \text{ if } \rho_m \begin{array}{l}\geq(1+\mu/\rho)(\rho_c\rho-f_{kk})>0\\ \leq(1+\mu/\rho)(\rho_c\rho-f_{kk})>0\end{array}. \tag{10b}$$

Apparently, if $\rho_m<0$, the level of welfare must be lower in a steady state because of a lower consumption level and a higher discount rate resulting from lower real balances. Alternatively, if $\rho_m>0$, the effect on the level of welfare is ambiguous. In the situation where a positive ρ_m is sufficiently large so the condition in (10b) is met, consumption increases so as to raise welfare although lower real balances reduce welfare.

4. Transaction Costs Model

In this Section, we turn to the TC model where money is introduced through a shopping time technology. Following Saving (1971) and Wang and Yip (1992), we assume that only consumption transactions are costly and money holdings facilitate transactions. Specifically, the transaction costs are assumed to take the form: $T(t)=T(c(t), m(t))$, where $T_c>0$, $T_{cc}>0$, $T_m<0<T_{mm}$, $T(0, m)=0$ and $T_{cm}\leq 0$. Thus, we assume the transaction costs are positive if there is positive

consumption. Moreover, the transaction costs function is convex in consumption and decreasing in real balances in a diminishing way. Finally, assumption $T_{cm} \leq 0$ indicates that higher real balances tend to lower the marginal transaction costs of consumption.

With transaction costs, the representative agent's budget constraint in (3) now becomes

$$\dot{a} = f(k) - c - \pi m + v - T(c,m). \tag{11}$$

The discounted lifetime utility is given by (1), but the felicity $u(c, m)$ in (1) is replaced by $u(c)$ and the discount rate $\rho(c, m)$ in (2) by $\rho(c)$. The Volterra derivative indicates that ρ is a function of c and the shadow price of the cumulative discount. The shadow price of the cumulative discount at time t in turn is the present value of the future discounted felicity at and after time t which is a function of $c(t)$. As a result, we replace $\rho(c(t), m(t))$ in (2) by $\rho(c(t))$.

In the Appendix, we have derived the representative agent's optimization problem. In equilibrium, while the money market clearance condition remains (5a), the goods market clearance condition in (5b) is modified as

$$\dot{k} = f(k) - c - T(c,m). \tag{12}$$

In the Appendix, we have simplified the equilibrium dynamic system into a 4x4 system characterized by c, m, k and λ. The way to determine the equilibrium path of key variables is also explained. We have taken a linear Taylor expansion of the system around a steady state. In order to assure a stable saddle in a steady state, the determinant of the Jacobean matrix of the linear system must be negative, denoted by $\Delta_{TC}<0$.

We are ready to analyze the money superneutrality in a steady state. In a steady state, $\dot{c} = \dot{m} = \dot{k} = \dot{\lambda} = 0$. The steady-state conditions are

$$f_k(k^*) = \rho(c^*), \tag{13a}$$

$$f(k^*) = c^* + T(c^*, m^*), \tag{13b}$$

$$\lambda^* = \frac{u(c^*)}{\rho(c^*)}, \tag{13c}$$

$$f_k(k^*) + \mu = -T_m, \tag{13d}$$

$$\xi^* = \rho(c^*)\theta^*. \tag{13e}$$

Note that compared the steady-state conditions here with those in the MIUF model, real balances affect a steady state via subjective time preferences in (7a), (7c) and (7e) but not in the corresponding (13a), (13c) and (13e). Rather, real balances affect the TC model here through real transaction costs in the CC condition (13b) and the optimal demand for real balances in (13d). Because of reducing real transaction costs, monetary expansion is usually not superneutral in the

TC model in a standard optimal growth model with elastic labor supply and an exogenous time preference. In our TC model with inelastic labor, money affects the KR rule in (13a), and thus capital, via the effect of consumption on time preference. As a result, the relationship between inflation and growth may be different.

To analyze the relationship between inflation and growth, the KR rule (13a) and the CC condition (13b) in the (k, c) plan are illustrated in Figures 3 and 4 with the steady state at E_0. When $\rho_c \leq 0$, the Correspondence Principle requires $f_{kk}(1+T_c)-\rho\rho_c<0$, which is assured under Assumption 2 and $T_c>0$. Both loci together yield

$$(f_{kk} - \tfrac{\rho_c \rho}{1+T_c})dk = -\tfrac{\rho_c T_m}{1+T_c} dm. \tag{14a}$$

Similar to our MIUF model in Section 3, changes in capital depend on changes in real balances, but the effect is now via reducing transaction costs $T_m<0$. If the time preference is independent of consumption, $\rho_c=0$. Then even if real balances reduce transaction costs, they only affect consumption in (13b) without spreading out the effect to (13a), thereby exerting no effect on capital. Capital is solely determined by (13a) in this special case. As a result, money is superneutral in a steady state. This is the result shared in the TC models in Wang and Yip (1992) and Zhang (2000) when their labor supply is inelastic. However, if consumption affects time preferences, then even if labor supply is inelastic, real balances remain exerting effects on capital.

The effect of a higher growth rate of money on real balances is obtained by differentiating (13d), with the use of (14a) and (13b),

$$\frac{dm}{d\mu} = \frac{m^* \rho(1+T_c)}{\Delta_{TC}\Omega}[f_{kk}(1+T_c) - \rho_c \rho] < 0, \tag{14b}$$

as $\Omega \equiv \frac{(u_{cc}-\lambda\rho_{cc})(1+T_c)}{u_c-\lambda\rho_c} - T_{cc} < 0$ and $\Delta_{TC}<0$.

It is clear that real balances are decreasing in the growth rate of nominal money supply even if $\rho_c=0$. As real balances decrease, the CC condition shifts downward (Figures 3 and 4). Obviously, even if $\rho_c=0$, consumption unambiguously decreases in a steady state because of higher transaction costs due to lower real balances. However, the effect on capital depends upon how the degree of impatience responds to lower consumption. In Figure 3 where $\rho_c \geq 0$, it is clear that capital increases. In contrast, in Figure 4 where $\rho_c \leq 0$, capital decreases.

[Insert Figures 3 and 4 here]

Next, the effect on capital in a steady state is obtained by substituting (14b) into (14a)

$$\frac{dk}{d\mu} = \frac{-\rho_c T_m}{f_{kk}(1+T_c)-\rho_c\rho}\frac{dm}{d\mu} = \frac{m^*\rho(1+T_c)}{\Delta_{TC}\Omega}(-\rho_c T_m) \mathop{\gtrless} 0 \text{ if } \rho_c \mathop{\gtrless} 0, \tag{14c}$$

and the effect on consumption is obtained by substituting (14b) and (14c) into (13b)

$$\frac{dc}{d\mu} = \tfrac{f_k}{1+T_c}\frac{dk}{d\mu} - \tfrac{T_m}{1+T_c}\frac{dm}{d\mu} = \frac{m\rho(1+T_c)}{\Delta_{TC}\Omega}(-f_{kk}T_m) < 0. \tag{14d}$$

Intuitively, lower real balances increase the transaction costs of consumption and thus discourage consumption. In an economy where an agent is *less patient* as he consumes more, $\rho_c>0$, lower consumption makes him more patient so he saves more. As a result, capital is higher in a steady state. Alternatively, when an individual is *more patient* as he consumes more, $\rho_c>0$, lower consumption makes him less patient. Consequently, capital is lower in a steady state.

Using (14b) and (14c), the change in wealth in a steady state is

$$\frac{da}{d\mu} == \frac{m^*\rho(1+T_c)}{\Delta_{TC}\Omega}[f_{kk}(1+T_c)-\rho_c(\rho+T_m)] \underset{\leq}{\geq} 0 \ \ if\ \rho_c T_m \underset{\geq}{\leq} f_{kk}(1+T_c)-\rho\rho_c < 0. \tag{15a}$$

The wealth may decrease or increase as a result of a higher growth rate of money. In the case where $\rho_c \leq 0$, as then $\rho_c T_m > 0$, real balances and capital both decrease and thus the amount of wealth decreases (Figure 4). Alternatively, in the case where $\rho_c \geq 0$, as $\rho_c T_m < 0$, the effect on wealth is ambiguous because real balances decrease while capital increases (Figure 3). In this case it is possible that wealth then increases in a steady state. This situation emerges when $\rho_c > f_{kk}(1+T_c)/(\rho+T_m) > 0$.[13] Under such a condition, the agent becomes sufficiently patient so an increase in capital is more than a decrease in real balances.

Finally, the effect on the lifetime utility in a steady state is negative (See Appendix)

$$\frac{dU}{d\mu} = \frac{\theta(1+T_c)}{\rho}\frac{dc}{d\mu} < 0. \tag{15b}$$

Intuitively, even if lower consumption may make people feel less impatient and creates an indirect offsetting effect through a possible lower degree of impatience in the case of $\rho_c \geq 0$, the direct effect of lower consumption on utilities apparently dominates. As a result, the lifetime utility is unambiguously lower in a steady state.

We now briefly compare the MIUF and TC models. Table 1 conveniently summarizes the comparative-static results for the two models with an endogenous time preference.

It is clear that if a time preference is affected by real balances in the MIUF model, money is not superneutral even without labor-leisure tradeoffs. The non-superneutrality result here is in line with Brock (1974) and his followers in models that rely on labor-leisure tradeoffs to create non superneutrality. The result differs from the superneutrality in Sidrauski (1967a, b) in models with an exogenous time preference. Our results reveal that a higher growth rate of money, and thus higher inflation, reduces capital, wealth, consumption and welfare in the long run in the case

[13] Notice $(\rho+T_m)<0$ as $\rho_c(\rho+T_m)<f_{kk}(1+T_c)<0$ and $\rho_c>0$ in this case.

where ρ_m<0, but increases capital and wealth in the case where ρ_m>0. The positive relationship between inflation and capital in the latter case features a Tobin effect in the sense of Tobin (1965). This positive relationship between inflation and capital is what has been argued and emphasized by Epstein and Hynes (1983) in the context of an endogenous time preference. The result in Epstein and Hynes (1983), however, is only one of the cases here that emerges only if real balances increase impatience. If real balances decrease impatience, the relationship between inflation and capital is negative in a steady state. Existing empirical evidence is in support of this case (Becker and Mulligan, 1994), but Epstein and Hynes (1983) neglected this result.

[Insert Table 1 here]

In the special case where ρ_m=0, real balances do not affect a time preference. In this case we obtain the money superneutrality even when consumption affects a time preference. In a MIUF model with an endogenous time preference, Hayakawa (1995) obtained the superneutrality only when there is a perfect complementarity between real balances and consumption.[14] In contrast, provided that a time preference is free from the effect of real balances, our MIUF model obtains the superneutrality under a very general utility function that allows for substitutability between real balances and consumption.

For the TC model with an endogenous time preference, results in Table 1 indicate that a higher growth rate of money, and thus higher inflation, unambiguously decreases real balances and consumption, and thus wealth, because of the transaction costs of consumption. This result is in line with the prediction in existing work with an exogenous time preference by Wang and Yip (1992) and Zhang (2000). Like these existing studies, money supply is not superneutral. However, depending on the response of time preferences to consumption, the relationship between inflation and capital here may be negative or positive in a steady state. The relationship is negative only when the degree of impatience is decreasing in consumption, ρ_c<0. When ρ_c>0 and the degree of impatience is sufficiently increasing in consumption, such a relationship is positive. Therefore, different from existing studies with an exogenous time preference, there is possibly a Tobin effect in the TC model with an endogenous time preference.

Following Wang and Yip (1992) and Zhang (2000), we now establish a qualitative equivalence between the MIUF and TC models. It is clear that it is impossible for a higher growth rate of money to increase all capital, wealth, consumption and welfare. Alternatively, if

[14] The setup in Hayakawa (1995) is intrinsically to impose a cash-in-advance constraint on consumption. In a similar model with an exogenous time preference, Asako (1983) relaxed the limiting assumption by proposing a utility function that is separable in real balances and consumption in order to obtain the superneutrality.

we only focus on the effect on the reallocation of assets, it is possible to identify a set of conditions so the Tobin effect emerges in both models. Indeed, a higher growth rate of money increases capital and wealth in both the MIUF and TC models when both conditions ρ_m>0 and ρ_c>0 hold and their magnitudes are sufficiently large. In Wang and Yip (1992, Table 1, p. 555) and Zhang (2000, Table 2, p. 10), there are no unified parameter restrictions so the Tobin effect emerges in their MIUF and TC models.

It is obvious there is a qualitative equivalence in terms of a reversed Tobin effect between the MIUF and TC models. When both conditions ρ_m<0 and ρ_c<0 hold, a higher growth rate of money decreases capital, wealth, consumption and welfare in both the MIUF and TC models. In Wang and Yip (1992, Table 1), a dominant consumption effect over a real balance effect is required in order to establish a qualitative equivalence in terms of a reversed Tobin effect between the MIUF and TC models. A similar condition is also required to establish a qualitative equivalence in terms of a reversed Tobin effect between the different TC models in Zhang (2000, Table 2). Moreover, labor-leisure tradeoffs are required in these two existing studies. In our model, there is the requirement of neither a dominant consumption effect nor labor-leisure tradeoffs. Rather, our equivalence result is established under endogenous time preferences.

Finally, we should mention the plausibility of the conditions for a qualitative equivalence in terms of the relationship between inflation and growth where ρ_c>0 and ρ_m>0 for a positive relationship and ρ_c<0 and ρ_m<0 for a negative relationship. In the former case, the required condition ρ_c>0 is consistent with Uzawa (1968), Lucas and Stokey (1984) and Obstfeld (1990). However, we find no existing evidence in support of the condition ρ_m>0. Thus, a Tobin effect in our model is less plausible. In the latter case, condition ρ_c<0 is consistent with that proposed by Fisher (1930), Koopmans (1960), Blanchard and Fischer (1989) and Barro and Sala-i-Martin (1995). Moreover, condition ρ_m<0 is consistent with the evidence in Becker and Mulligan (1994). In view of a support in favor of ρ_m<0, it is more plausible that both the MIUF and TC models are qualitatively equivalent in terms of a negative relationship between inflation and growth in a steady state.

5 Concluding Remarks

This paper revisits the issue of the relationship between inflation and growth in the long run. Different from the work by Brock (1974), Wang and Yip (1992) and Zhang (2000), we do not rely on a labor-leisure tradeoff in order to establish the non superneutrality of money. We focus on the role of an endogenous time preference upon the demand for assets between real cash balances and capital in an optimal growth model between the MIUF and TC approaches. Consideration of an endogenous time preference influences the marginal rate of substitution between consumption now and in the future and thus changes capital accumulation. As a result, we find a qualitative equivalence between the MIUF and TC models is easy to establish without relying on labor-leisure tradeoffs and a dominant consumption effect. Our results are in sharp contrast to those obtained in existing models with an exogenous time preference.

In the MIUF model, we use an endogenous rate of time preference that depends not only on consumption flows but also on real balances. In the TC model, we use an endogenous rate of time preference that depends only on consumption flows. Both setups are consistent with endogenous time preference when applying the Volterra derivative. Even in the absence of elastic labor, an endogenous time preference easily spreads the effect of real balances over to the optimal demand for capital and thus exerts an effect on capital in a steady state.

In these two models, a higher inflation always leads to lower real balances. However, the effect on capital and real assets depends on the degree of impatience in response to consumption and real balances. Under increasing impatience in consumption, as proposed in Uzawa (1968) and others, and increasing impatience in real balances, we find higher capital and wealth in association with a higher inflation in both the MIUF and TC models. As a result, the relationship between inflation and growth is positive in a steady state, as was argued in Tobin (1965). However, there is no evidence pointing to increasing impatience in real balances. Alternatively, under decreasing impatience in both consumption and real balances, a higher inflation reduces capital and wealth in both the MIUF and TC models, thus a reverse Tobin effect. Decreasing impatience in consumption is in line with Fisher (1930) and his followers, and decreasing impatience in real balances is also consistent with the evidence in Becker and Mulligan (1994). Therefore, in this plausible case the MIUF and TC models are qualitatively equivalent in terms of a negative relationship between inflation and growth in a steady state.

Finally, we only consider a time preference affected by individual consumption and individual real balances in the current model. Alternatively, time preferences may be affected by average (social) consumption and average real balances in an economy; thus there are admiration and zealousy effects (e.g., Meng, 2006). Moreover, the subjective discount may be affected by either own past consumption (individual habits, e.g., Chen, 2006) or average past consumption (social habits, e.g., Alvarez-Cuadrado, et al 2004); thus there are the effects of catching up with the Joneses. It may be interesting to see how consideration of different ways of formulating endogenous time preferences may affect the agent's saving behavior and thus capital formation, an avenue for future research.

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Appendix to "Inflation and Growth: Impatience and a Qualitative Equivalence"

Been-Lon Chen, Mei Hsu and Chia-Hui Lu

Appendix 1 Conditions to Assure a Stable Saddle in the MIUF Model

To assure a saddle stable steady state, if we take Taylor's expansion of system (4d), (5b), (6a) and (6b) in the neighborhood of the steady state, we obtain

$$\begin{bmatrix} \dot{c} \\ \dot{m} \\ \dot{k} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & J_{24} \\ -1 & 0 & f_k & 0 \\ \lambda\rho_c - u_c & \lambda\rho_m - u_m & 0 & \rho \end{bmatrix} \begin{bmatrix} c-c^* \\ m-m^* \\ k-k^* \\ \lambda-\lambda^* \end{bmatrix}, \tag{A1}$$

where $J_{11} = -\frac{u_{cm}-\lambda\rho_{cm}}{u_{cc}-\lambda\rho_{cc}} J_{21}$,

$$J_{12} = \frac{\rho_m(u_c-\lambda\rho_c)}{u_{cc}-\lambda\rho_{cc}} + \frac{\rho_c(\lambda\rho_m-u_m)}{u_{cc}-\lambda\rho_{cc}} - \frac{u_{cm}-\lambda\rho_{cm}}{u_{cc}-\lambda\rho_{cc}} J_{22},$$

$$J_{13} = \frac{-f_{kk}(u_c-\lambda\rho_c)}{u_{cc}-\lambda\rho_{cc}} - \frac{u_{cm}-\lambda\rho_{cm}}{u_{cc}-\lambda\rho_{cc}} J_{23},$$

$$J_{14} = \frac{\rho_c\rho}{u_{cc}-\lambda\rho_{cc}} - \frac{u_{cm}-\lambda\rho_{cm}}{u_{cc}-\lambda\rho_{cc}} J_{24},$$

$$J_{21} = -\left[\frac{u_{cm}-\lambda\rho_{cm}-(\rho+\mu)(u_{cc}-\lambda\rho_{cc})}{u_c-\lambda\rho_c}\right] m^*,$$

$$J_{22} = -\left[\frac{u_{mm}-\lambda\rho_{mm}-(\rho+\mu)(u_{cm}-\lambda\rho_{cm})}{u_c-\lambda\rho_c}\right] m^*$$

$$J_{24} = -\left[\frac{-\rho_m+(\rho+\mu)\rho_c}{u_c-\lambda\rho_c}\right] m^*,$$

$$J_{23} = f_{kk} m^* < 0.$$

In the 4x4 dynamic equilibrium system, c, m and λ are all control variables whose initial values are not predetermined. Their value can jump instantaneously. The variable k is a state variable whose value is initially predetermined. As a result, the unique steady state is a saddle if the number of eigenvalues with negative real parts is one. This situation is possible only if the determinant of the Jacobean in (A1) is negative. Denote as Δ the determinant of the Jacobean. Then, it is required to impose

$$\Delta = \left(\frac{m}{u_{cc}-\lambda\rho_{cc}}\right) \left\{ \begin{array}{l} f_{kk}[(\rho+\mu)^2\rho_c\theta - \rho\rho_m\theta - \rho_m(\rho+\mu)\theta + \rho(u_{mm}-\lambda\rho_{mm}) - \rho(\rho+\mu)(u_{cm}-\lambda\rho_{cm})] \\ +\rho^2\{[\rho_c(\rho+\mu)+\rho_m](u_{cm}-\lambda\rho_{cm}) - \rho_c(u_{mm}-\lambda\rho_{mm}) - \rho_m(\rho+\mu)(u_{cc}-\lambda\rho_{cc})\} \\ -\rho\theta[\rho_m-\rho_c(\rho+\mu)]^2 \end{array} \right\} < 0.$$

Appendix 2 A Cash-in-advance Constraint on Consumption

The Hamiltonian function associated to the representative agent's optimization program in an economy with the cash-in-advance constraint on consumption ($c \leq m$) is

$$H(c,k,m,\theta,\lambda,\varphi,\xi) = X\{u(c,m) + \theta[f(k) - c - \pi m + v] + \xi(a - m - k) + \varphi(m - c) - \lambda\rho(c,m)\},$$

where φ is the Lagrange multiplier associated with the cash-in-advance constraint.

Applying the maximum principle, the necessary conditions are

$$\theta = u_c(c,m) - \lambda\rho_c(c,m) - \varphi, \tag{A2a}$$

$$\theta f_k(k) = \xi, \tag{A2b}$$

$$-\theta\pi + \varphi + u_m(c,m) - \lambda\rho_m(c,m) = \xi, \tag{A2c}$$

$$\dot{\lambda} = -u(c,m) + \lambda\rho(c,m), \tag{A2d}$$

$$\dot{\theta} = \rho(c,m)\theta - \xi, \tag{A2e}$$

along with the transversality conditions $\lim_{t\to\infty}\lambda(t)X(t) = 0$ and $\lim_{t\to\infty}\theta(t)X(t)a(t) = 0$.

The steady-state conditions, with the binding cash-in-advance constraint, are

$$m^* = c^*, \tag{A3a}$$

$$f(k^*) = c^*, \tag{A3b}$$

$$f_k(k^*) = \rho(c^*,c^*), \tag{A3c}$$

$$\lambda^* = \frac{u(c^*,c^*)}{\rho(c^*,c^*)}, \tag{A3d}$$

$$f_k(k^*) + \mu = \frac{u_m(c^*,c^*)/u(c^*,c^*) - \rho_m(c^*,c^*)/\rho(c^*,c^*) + \phi^*/u(c^*,c^*)}{u_c(c^*,c^*)/u(c^*,c^*) - \rho_c(c^*,c^*)/\rho(c^*,c^*) - \phi^*/u(c^*,c^*)}, \tag{A3e}$$

$$\theta^* = \frac{\varphi^* - \lambda^*\rho_m(c^*,c^*)}{f_k(k^*) + \mu}, \tag{A3f}$$

$$\xi^* = \rho(c^*,c^*)\theta^*. \tag{A3g}$$

Apparently, with a binding cash-in-advance constraint in (A3a), the CC condition (A3b) and the KR rule (A3c) can determine c^* and k^* in a steady state without relying on other steady-state conditions. Once consumption is determined by (A3b) and (A3c), the real shadow price of the cumulative discount, λ^*, is determined by (A3d) independent of other variables. As a result, the economy is dichotomized in the fashion as if ρ_m=0. Thus, when the growth rate of nominal money supply increases, it affects the shadow price of the cash constraint, φ^*, of capital, θ^*, and of the assets, ξ^*, through (A3e)-(A3g). As a result, the growth rate of nominal money supply only affects the prices, without exerting any effects on capital, consumption and real balances. Indeed, we have shown $\frac{d\varphi}{d\mu}=\frac{\theta}{1+f_k+\mu}>0$, $\frac{d\theta}{d\mu}=\frac{\theta}{\rho+\mu}\frac{-(f_k+\mu)}{1+f_k+\mu}<0$ and $\frac{d\xi}{d\mu}=\frac{\theta\rho}{\rho+\mu}\frac{(f_k+\mu)}{1+f_k+\mu}>0$, while $\frac{dm}{d\mu}=\frac{dc}{d\mu}=\frac{dk}{d\mu}=\frac{d\lambda}{d\mu}=0$.

Appendix 3 Derivation of (8b)

Differentiating (7d), together (7b), yields

$$\begin{aligned}&\{f_{kk}\theta-\rho[u_{mc}-\lambda\rho_{mc}-\rho_m\tfrac{\rho u_c-u\rho_c}{\rho^2}]+\rho(\rho+\mu)[u_{cc}-\lambda\rho_{cc}-\rho_c\tfrac{\rho u_c-u\rho_c}{\rho^2}]\}\tfrac{dk}{d\mu}=\\&\{[u_{mm}-\lambda\rho_{mm}-\rho_m\tfrac{\rho u_m-u\rho_m}{\rho^2}]+(\rho+\mu)[u_{cm}-\lambda\rho_{cm}-\rho_c\tfrac{\rho u_m-u\rho_m}{\rho^2}]\}\tfrac{dm}{d\mu}-\theta.\end{aligned}\tag{A4}$$

Next, (4a), (4b), (4c) and (7a) lead to

$$\tfrac{\rho u_c-u\rho_c}{\rho^2}=\tfrac{\theta}{\rho}\ \text{and}\ \tfrac{\rho u_m-u\rho_m}{\rho^2}=\tfrac{\theta}{\rho}(\mu+\rho).$$

Thus, (A4) becomes

$$\begin{aligned}&\rho\{f_{kk}\tfrac{\theta}{\rho}+[(\rho+\mu)(u_{cc}-\lambda\rho_{cc})-(u_{mc}-\lambda\rho_{mc})]+\tfrac{\theta}{\rho}[\rho_m-(\rho+\mu)\rho_c]\}\tfrac{dk}{d\mu}+\\&\{[(\rho+\mu)(u_{cm}-\lambda\rho_{cm})-(u_{mm}-\lambda\rho_{mm})]+\tfrac{\theta}{\rho}[\rho_m-(\rho+\mu)\rho_c](\rho+\mu)\}\tfrac{dm}{d\mu}=-\theta,\end{aligned}$$

which is rewritten as

$$\begin{aligned}&\rho\{f_{kk}\tfrac{\theta}{\rho}+\tfrac{(\rho+\mu)(u_{cc}-\lambda\rho_{cc})-(u_{mc}-\lambda\rho_{mc})}{\theta}+\tfrac{\theta}{\rho}\tfrac{\rho_m-(\rho+\mu)\rho_c}{\theta}\}\tfrac{dk}{d\mu}+\\&\{\tfrac{(\rho+\mu)(u_{cm}-\lambda\rho_{cm})-(u_{mm}-\lambda\rho_{mm})}{\theta}+\tfrac{\theta}{\rho}(\rho+\mu)\tfrac{\rho_m-(\rho+\mu)\rho_c}{\theta}\}\tfrac{dm}{d\mu}=-1,\end{aligned}$$

and finally, as

$$\rho\{f_{kk}\tfrac{\theta}{\rho}+J_{21}\tfrac{1}{m}+\tfrac{\theta}{\rho}J_{24}\tfrac{1}{m}\}\tfrac{dk}{d\mu}+\{J_{22}\tfrac{1}{m}+\tfrac{\theta}{\rho}(\rho+\mu)J_{24}\tfrac{1}{m}\}\tfrac{dm}{d\mu}=-1.\tag{A5}$$

Appendix 4 Welfare in the MIUF Model in a Steady State

Using (1) and (2), the level of welfare in a steady state is

$$U=\frac{u(c^*,m^*)}{\rho(c^*,m^*)}.\tag{A6}$$

Differentiating (A6) with respect to c^*, m^* and μ yields

$$\frac{dU}{d\mu}=\frac{u_c-\lambda\rho_c}{\rho}\frac{dc}{d\mu}+\frac{u_m-\lambda\rho_m}{\rho}\frac{dm}{d\mu}.$$

Substituting (9a)-(9c) and using (7d), we rewrite the above expression as

$$\frac{dU}{d\mu}=(u_c-\lambda\rho_c)(\frac{\rho_m}{f_{kk}-\rho_c\rho}+1+\frac{\mu}{\rho})\frac{dm}{d\mu}.\tag{A7}$$

Appendix 5 Optimization and Equilibrium Conditions in the TC Model

The Hamiltonian associated to the representative agent's optimization program is

$$H(c,k,m,\theta,\lambda,\xi)=X\{u(c)+\theta[f(k)-c-\pi m+\nu-T(c,m)]-\lambda\rho(c)+\xi(a-m-k)\}.$$

Applying the maximum principle, the necessary conditions are

$$\theta=u_c(c)-\lambda\rho_c(c)-\theta T_c(c,m), \tag{A8a}$$

$$\theta f_k(k)=\xi\,, \tag{A8b}$$

$$\theta\pi+\xi=-\theta T_m(c,m), \tag{A8c}$$

$$\dot{\lambda}=-u(c)+\lambda\rho(c), \tag{A8d}$$

$$\dot{\theta}=\rho(c)\theta-\xi, \tag{A8e}$$

along with the transversality conditions $\lim_{t\to\infty}\lambda(t)X(t)=0$ and $\lim_{t\to\infty}\theta(t)X(t)a(t)=0$.

All the above optimal conditions are similar to those in (4a-4e) in the MIUF model, except that the marginal utility of consumption in (A8a) is net of transaction costs, and the marginal utility of real balances in (A8c) includes the facilitation of transactions.

The equilibrium is constituted by (A8a-e), (5a) and (12) that determines $\{c, m, k, \lambda, \theta, \xi, \pi\}$. To simplify the equilibrium conditions, (5a) and (A8a-c) lead to

$$\dot{m}=\left(\mu+f_k+T_m\right)m. \tag{A9a}$$

Moreover, differentiating (A8a) and using (A8b) yields

$$\dot{c}=\tfrac{1}{\Omega}\left[(1+T_c)(\rho-f_k)+T_{cm}\dot{m}+\tfrac{\rho_c(1+T_c)}{u_c-\lambda\rho_c}\dot{\lambda}\right], \tag{A9b}$$

where $\Omega\equiv\frac{(u_{cc}-\lambda\rho_{cc})(1+T_c)}{u_c-\lambda\rho_c}-T_{cc}<0.$

Thus, (A8d), (12) and (A9a-b) constitute a simplified 4x4 dynamic equilibrium system. These four equations solve $\{c, m, k, \lambda\}$, and the other three variables are determined by using (A8a), (A8c) and (A8e).

If we take a linear Taylor expansion of dynamic system (A8d), (12) and (A9a-b) around the unique steady state $\{c^*, k^*, m^*, \lambda^*\}$, we obtain the following:

$$\begin{bmatrix}\dot{c}\\ \dot{m}\\ \dot{k}\\ \dot{\lambda}\end{bmatrix}=\begin{bmatrix}\tilde{J}_{11} & \tilde{J}_{12} & \tilde{J}_{13} & \tilde{J}_{14}\\ T_{cm}m^* & T_{mm}m^* & f_{kk}m^* & 0\\ -1-T_c & -T_m & f_k & 0\\ \lambda\rho_c-u_c & 0 & 0 & \rho\end{bmatrix}\begin{bmatrix}c-c^*\\ m-m^*\\ k-k^*\\ \lambda-\lambda^*\end{bmatrix}, \tag{A10}$$

where $\tilde{J}_{11}=\frac{1}{\Omega}T_{cm}^2m^*,$

$\tilde{J}_{12}=\frac{T_{cm}}{\Omega}T_{mm}m^*,$

$\tilde{J}_{13} = \frac{f_{kk}}{\Omega}[-(1+T_c)+T_{cm}m^*]$,

$\tilde{J}_{14} = \frac{\rho_c(1+T_c)\rho}{(u_c-\lambda\rho_c)\Omega}$.

In the dynamic system, only capital is initially predetermined. As a result, to ensure a stable saddle in a steady state, it is required that the determinant is negative, denoted by $\Delta_{TC}<0$.

Appendix 6 Welfare in the TC Model in a Steady State

The level of utility in the steady state is

$$U = \frac{u(c^*)}{\rho(c^*)}. \tag{A11}$$

Differentiating (A11) with respect to c^* and μ yields

$$\frac{dU}{d\mu} = \frac{1}{\rho}(u_c - \frac{u}{\rho}\rho_c)\frac{dc}{d\mu}.$$

Finally, using (A8a), we rewrite the above expression as

$$\frac{dU}{d\mu} = \frac{\theta(1+T_c)}{\rho}\frac{dc}{d\mu} < 0. \tag{A12}$$

Table 1. Comparative-static results of a higher growth rate of money

		m	K	a	C	U
MIUF	$\rho_m > 0$	−	+	+ [a]	+	+ [b]
	$\rho_m < 0$	−	−	−	−	−
TC	$\rho_c > 0$	−	+	+ [c]	−	−
	$\rho_c < 0$	−	−	−	−	−

Note. [a] $\rho_m > \rho\rho_c - f_{kk} > 0$,
[b] $\rho_m > (1+\mu/\rho)(\rho\rho_c - f_{kk}) > 0$,
[c] $\rho_c > f_{kk}(1+T_c)/(\rho+T_m) > 0$.

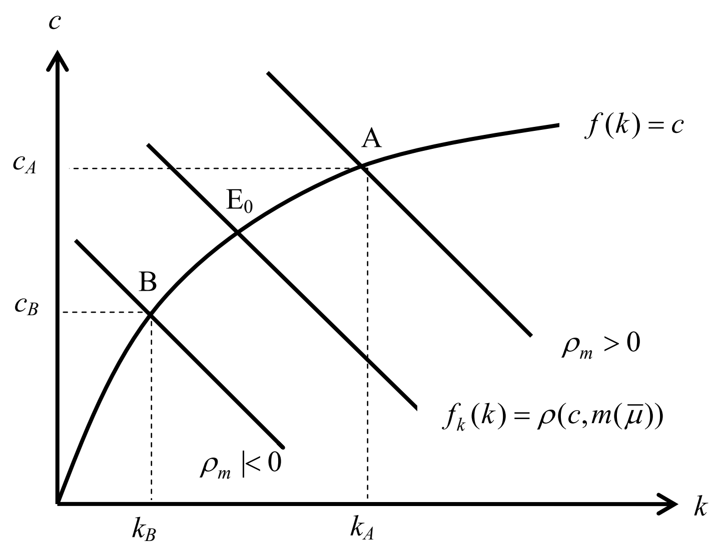

Figure 1. A higher μ and thus $dm < 0$ in the money-in-utility-function model: case $\rho_c \geq 0$.

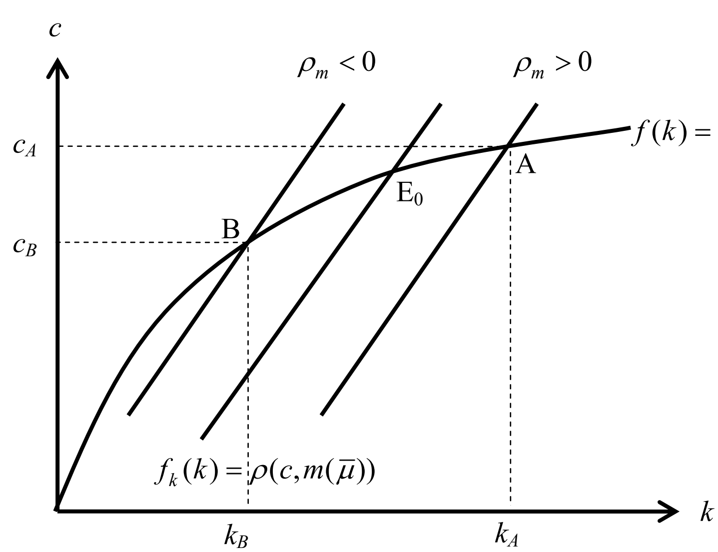

Figure 2. A higher μ and thus $dm < 0$ in the money-in-utility-function model: case $\rho_c \leq 0$.

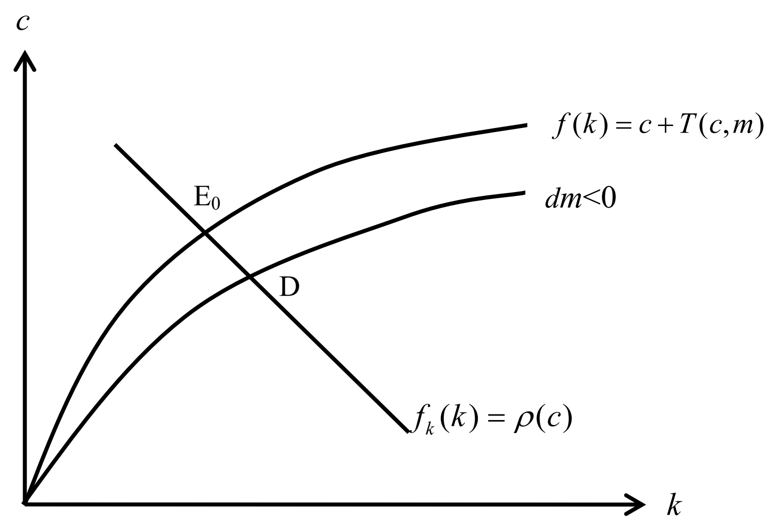

Figure 3. A higher μ in the transaction costs model: case $\rho_c \geq 0$.

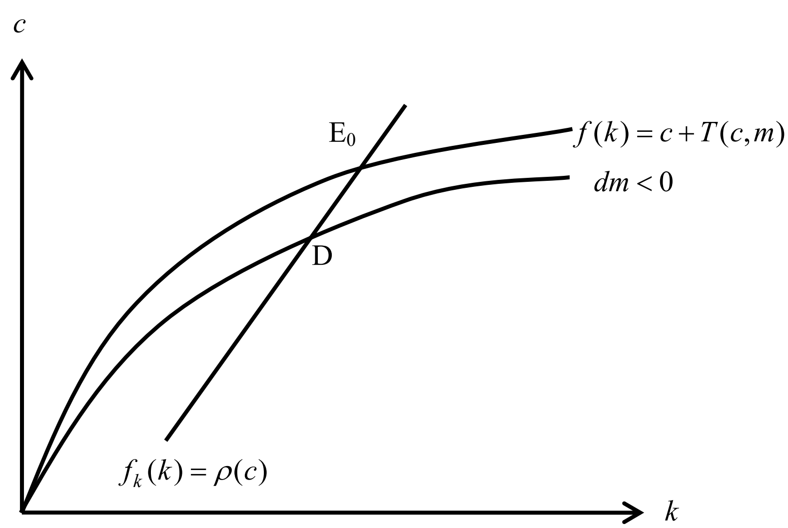

Figure 4. A higher μ in the transaction costs model: case $\rho_c \leq 0$.