Who Prefers Minimum or Maximum Price Policies?: Truncated Increases in Risk and Linear Payoffs*

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Abstract

This paper introduces the new type of a 'truncated increase in risk' (TIR) which is an increase in risk in the Kroll, Leshno, Levy and Spector (1995; called K-L-L-S) sense, based on a concept of an increase in risk with truncated distributions that reveals real economic phenomena such as guaranteed minimum or imposed maximum prices. We also show that the comparative static analysis for the TIR is applied to all risk-averse decision-makers with prudence.

Keywords: Rothschild-Stiglitz increases in risk; Kroll, Leshno, Levy and Spector (K-L-L-S) increases in risk; truncated increase in risk

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1. Introduction

An important comparative static question in the study of economic models under uncertainty is how changes in the distribution of the random variable in the decision model affect the level of the choice variable selected by a decision-maker. Since Rothschild and Stiglitz (1970, 1971) developed a definition of an increase in risk, several researchers have derived intuitively appealing comparative statics results for the risk-averse decision-makers by restricting the set of Rothschild and Stiglitz (R-S) increases in risk. On the other hand, Kroll, Leshno, Levy and Spector (1995; called K-L-L-S) defined a special 'increase in risk' and we call it an 'K-L-L-S increase in risk' to contrast an R-S increase in risk. Note that K-L-L-S increases in risk extend the R-S definition of risk to a larger set of cumulative distribution functions (CDFs) that could not be classified as 'more risky' before.

Eeckhoudt and Hansen (1980) introduced the concept of a 'min-max truncation' or a 'mean preserving truncation' which is the subset of the set of R-S increases in risk and showed that it led to an increase in output for a risk-averse firm. It can be viewed as the existence of guaranteed minimum and imposed maximum prices in real economic phenomena. When we consider minimum or maximum price policies separately in such a way that the distributional change in a mean-preserving increase in risk is obtained, who prefers what policy?

In this paper we define the subset of K-L-L-S increases in risk with truncated distributions by imposing the restrictions on the changes in CDFs and call it a 'truncated increase in risk' (TIR). We also provide the interesting comparative static result for the SIR_K order that causes risk-averse decision-makers with prudence to adjust the choice variable in the same direction when faced with a given shift in a random parameter.

This paper is organized as follows. In section 2, we present the new definition of a truncated increase in risk (TIR) and give a graphical example for the TIR. In section 3, we carry out the comparative static analysis for the TIR order and indicate how our general result applies to a specific economic model. Finally, section 4 contains concluding remarks.

2. Definitions

By analogy to the R-S definition of the mean preserving spread (MPS), Kroll et al. (1995) introduced a new concept of probability mass shifts and specified the conditions on the a 'mean preserving spread-anitspread' (MPSA) functions that enabled the classification of one random variable as 'more risky' than another random variable. We call this new definition an 'increase in risk in the K-L-L-S sense' and follow the formal definition provided by Ryu and Kim (2005):

***Definition 1*:** $G(x)$ is said to be riskier than $F(x)$ in the K-L-L-S sense if and only if

(a) $\int_a^b [G(x) - F(x)]dx = 0$

(b) $\int_a^s \int_a^t [G(x) - F(x)]dxdt \geq 0$ for all $s \in [a, b]$.

Condition (a) implies that two distributions have equal means. Condition (b) includes that a MPSA function satisfies the TSD criterion. These conditions imply that an increase in risk in the K-L-L-S sense is an TSD change with equal means. Observe that, for random variables with equal means, Definition 1 is equivalent to the TSD rule.

We now propose a type of risk increases that is a subset of K-L-L-S increases in risk and specify a 'truncated increase in risk' (TIR) which is defined by imposing restrictions on the difference between the two cumulative of CDFs (C-CDFs). We assume that the supports of $F(x)$ and $G(x)$ are located in the interval $[x_2, x_4]$ and $[x_1, x_3]$, respectively, where $x_1 \leq x_2 \leq x_3 \leq x_4$. Note also that $F(x)$ is the cumulative distribution function with minimum price and $G(x)$ with maximum price. Further we define that $\hat{F}(x) = \int_{x_1}^{x} F(t)\,dt$ and $\hat{G}(x) = \int_{x_1}^{x} G(t)\,dt$.

***Definition 2*:** $G(x)$ represents a strong increase in risk in the K-L-L-S sense with respect to $F(x)$ (denoted by *G* TIR *F*) if

(a) $\int_{x_1}^{x_4} [G(x) - F(x)]dx = 0$,

(b) $\int_{x_1}^{s}\left[\hat{G}(x)-\hat{F}(x)\right]dx \geq 0$ for all $s \in [x_1, x_4]$,

(c) There exists a unique point $m \in [x_2, x_3]$ such that $\hat{G}(m)-\hat{F}(m)=$ $\int_{x_1}^{m}G(x)dx - \int_{x_1}^{m}F(x)dx = 0$

(d) $\hat{G}(x)-\hat{F}(x)$ is non-increasing on (x_2, x_3).

Condition (a) implies that the two distributions have the equal mean. Condition (b) expresses the third-degree stochastic dominance (TSD) between $F(x)$ and $G(x)$. These conditions are sufficient for $G(x)$ to represent an increase in risk in the K-L-L-S sense from $F(x)$. Condition (c) implies that $\hat{G}(x)$ and $\hat{F}(x)$ cross only once at the point *m*. Condition (d) states that $G(x) \leq F(x)$, that is, $G(x)$ never exceeds $F(x)$ in the interval $[x_2, x_3]$. This implies that $G \leq F$. This is the added condition which allows general statements to be made concerning the effect of an TIR on the choice made by a risk-averse decision-maker with $u''' \geq 0$. Figure 1 illustrates an example of a truncated increase in risk (TIR). The transition from $G(x)$ to $F(x)$ implies that the risk transformation extends the set of possible outcomes to the right but truncates it to the left.

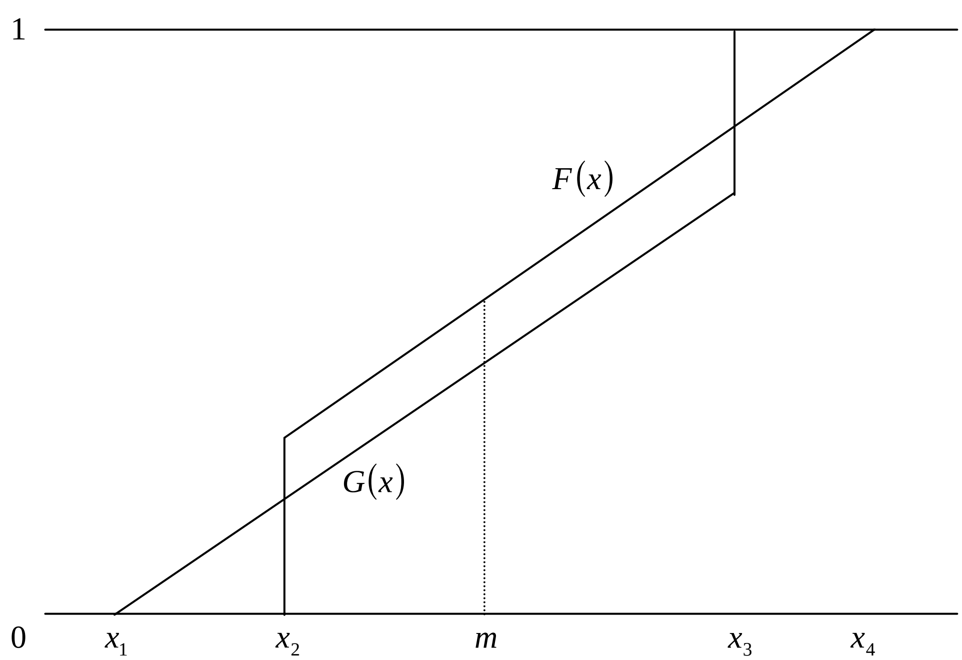

Fig. 1. G SIR_K F.

3. Comparative Static Analysis

Before we provide a general comparative static statement regarding the $SIR_{K,}$ a decision model is specified. We use the general decision model previously employed by Krause (1979) and Katz (1981) in their work. In this paper the economic agent is assumed to choose α to maximize $E(u[z(x,\alpha)])$, where x is a given random variable and α is a choice variable. That is, he selects α in order to maximize expected utility, where utility depends only on the outcome variable $z(x,\alpha)$, which is a scalar valued function of the choice variable and the random variable. The function $z(x,\alpha)$ is assumed to be linear in the choice variable α and the random parameter x. We assume that utility function $u(z)$ is thrice differentiable with respect to its argument with $u'(z) > 0$, $u''(z) < 0$, and $u'''(z) \geq 0$; thus, the decision maker is a risk averter with $u'''(z) \geq 0$. In order to focus on interior solutions to the maximization problem, it is assumed that $z_\alpha(x,\alpha) = 0$ is satisfied for some finite α for all relevant values of x. Faced by the CDF of the random variable, $F(x)$, the first-order condition defining the optimal value α_F for α is written as

$$\int_{x_1}^{x_4} u'[z(x,\alpha)]\, z_\alpha(x,\alpha)\, dF(x) = 0 \tag{1}$$

Given the assumptions, this condition is necessary and sufficient for this economic problem. In order to see the result that $\alpha_F \geq \alpha_G$ for a specified change in CDF from F to G, it is sufficient to show that

$$Q(\alpha_F) = \int_{x_1}^{x_4} u'[z(x,\alpha_F)]\, z_\alpha(x,\alpha_F)\, d[G(x) - F(x)] \leq 0 . \tag{2}$$

Before proposing and proving Theorem, the following result from Yitzhaki (1983) is needed.

***Lemma*:** Let $s(x)$ be a function defined on $[a,b]$ and $\int_a^y s(x)\,dx \geq 0$ for all $y \in [a,b]$. If another function $l(x)$ is non-negative and non-increasing for all $x \in [a,b]$, then $\int_a^y s(x)\, l(x)\, dx \geq 0$ for all $y \in [a,b]$.

***Theorem*:** If G TIR F and $z_{xx}=0$, then $\alpha_F \geq \alpha_G$ for risk-averse decision-makers with $u''' \geq 0$.

***Proof*:** From (2), $Q(\alpha_F)$ can be written as

$$Q(\alpha_F)=\int_{x_1}^{x_4} u'(z)\, z_\alpha(x,\alpha_F)\, dH(x) \tag{3}$$

where $H(x)=G(x)-F(x)$. Since $H(x_1)=H(x_4)=0$, integration by parts of (3) yields

$$Q(\alpha_F)=-\int_{x_1}^{x_4}\{u'(z)\, z_{\alpha x}(x,\alpha_F)+u''(z)\, z_x z_\alpha(x,\alpha_F)\}H(x)\, dx$$

$$=-\int_{x_1}^{x_4} u'(z)\, z_{\alpha x}(x,\alpha_F)H(x)\, dx-\int_{x_1}^{x_4} u''(z)\, z_x z_\alpha(x,\alpha_F)H(x)\, dx. \tag{4}$$

First, let's consider the first term in (4). Integration by parts of the first term in (4) yields

$$-u'(z)z_{\alpha x}(x,\alpha_F)\hat{H}(x)\Big|_{x=x_1}^{x_4}+\int_{x_1}^{x_4} u''(z)z_x z_{\alpha x}(x,\alpha_F)\, \hat{H}(x)\, dx. \tag{5}$$

The first term of (5) is zero because of the condition (a) in Definition 2. Integration by parts of the second term in (5) yields

$$u''(z)\, z_x z_{\alpha x}\int_{x_1}^{x} \hat{H}(t)\, dt\Big|_{x=x_1}^{x_4}-\int_{x_1}^{x_4} u'''(z)\, z_x^2 z_{\alpha x}\int_{x_1}^{x} \hat{H}(t)\, dtdx. \tag{6}$$

The sign of (6) is non-positive because of the condition (b) in Definition 2 and assumptions made about z_α.

Second, let's consider the second term in (4). Integration by parts of the second term in (4) yields

$$-u''(z)\, z_x\int_{x_1}^{y} z_\alpha(x,\alpha_F)H(x)\, d\, x\Big|_{y=x_1}^{x_4}+\int_{x_1}^{x_4} u'''(z)\, z_x^2\int_{x_1}^{y} z_\alpha(x,\alpha_F)H(x)\, dxdy. \tag{7}$$

A sufficient condition for (7) to be non-positive is

$$\int_{x_1}^{y} z_\alpha(x,\alpha_F)H(x)\, dx \leq 0 \quad \text{for all } y\in[x_1,x_4]. \tag{8}$$

Integration by parts of (8) yields

$$z_\alpha(x,\alpha_F)\hat{H}(x)\Big|_{x=x_1}^{y}-\int_{x_1}^{y} z_{\alpha x}(x,\alpha_F)\hat{H}(x)\, dx. \tag{9}$$

Using Lemma shows that the integral in (9) is non-negative because of the condition (b) in Definition 2 and assumptions made about z_α. The first term in (9) is equal to

$$z_{\alpha}(x,\alpha_F)\hat{H}(y) \tag{10}$$

which is zero when $y = x_F$ where x_F is the point of x when $z_{\alpha}(x,\alpha_F) = 0$. Hence, (8) is satisfied at x_F. Taking into account Definition 2, the sign of (10) is non-positive for y belonging to $[x_1, x_2]$ and because of $z_{\alpha}(y,\alpha_F)$ is non-positive for $y \le x_F$, (8) is also satisfied for y belonging to (x_2, x_F). Indeed,

$$\int_{x_1}^{y} z_{\alpha}(x,\alpha_F)H(x)\,dx = \int_{x_1}^{x_F} z_{\alpha}(x,\alpha)H(x)\,dx - \int_{y}^{x_F} z_{\alpha}(x,\alpha_F)H(x)\,dx\,. \tag{11}$$

Clearly, the first term on the right-hand-side of (11) is non-positive, as shown above, and the second integral is positive because of the condition (d) in Definition 2 and the assumptions made about z_{α}.

For y in (x_F, x_3) one gets

$$\int_{x_1}^{y} z_{\alpha}(x,\alpha_F)H(x)\,dx = \int_{x_1}^{x_F} z_{\alpha}(x,\alpha_F)H(x)\,dx + \int_{x_F}^{y} z_{\alpha}(x,\alpha_F)H(x)\,dx\,. \tag{12}$$

Because of the condition (d) in Definition 2, the sign of (12) is non-positive. Finally, for $y \in [x_3, x_4]$, the sign of (10) is non-positive from Definition 2. Hence, (8) holds for all $y \in [x_1, x_4]$.

Q.E.D.

Our result in Theorem shows that all risk-averse decision-makers with non-negative third derivative of utility functions $u''' \ge 0$ reduce the choice variable when the cumulative distribution function with minimum price changes to that with maximum price.

4. Concluding Remarks

This paper introduces the notion of a ‘truncated increase in risk’ (TIR) which is defined by imposing restrictions on the difference between the two cumulative of CDFs. The comparative static analysis for TIR is applied to all risk-averse decision-makers with non-negative third derivative of utility functions $u''' \ge 0$. An TIR shift can be interpreted as the mean-preserving K-L-L-S increase in risk such that the distribution of a certain prospect with minimum price changes to another with maximum.

References

Eeckhoudt, L. and Hansen, P. (1980): “Minimum and Maximum Prices, Uncertainty, and the Theory of the Competitive Firm.” *American Economic Review* Vol. 70(No. 5): 1064-1068.

Katz, E. (1981): “A Note on a Comparative Statics Theorem for Choice under Risk.” *Journal of Economic Theory* 25(No. 2): 318-319.

Kraus, M. (1979): “A Comparative Statics Theorem for Choice under Risk.” *Journal of Economic Theory* 21(No. 3): 510-517.

Kroll, Y., Leshno, M., Levy, H.. and Spector, Y. (1995): “Increasing Risk, Decreasing Absolute Risk Aversion and Diversification.” *Journal of Mathematical Economics* 24(No. 6): 537-556.

Rothschild, M., and Stiglitz, J. (1970): “Increasing Risk: I. A Definition.” *Journal of Economic Theory* 2(No. 3): 225-243.

Rothschild, M., and Stiglitz, J. (1971): “Increasing Risk: II. Its Economic Consequences.” *Journal of Economic Theory* 3(No. 1): 66-84.

Ryu, S., and Kim, I. (2005): “Portfolio Choice for Increases in Risk and Prudence Revisited.” *Journal of Economics* 86(No. 3): 293-300.

Yitzhaki, S. (1983): “On an Extension of the Gini Inequality Index.” *International Economic Review* 24(No. 3): 617-628.