Long-Run Welfare Implication of Helicopter Money in Aging Economies

Kiyohiko G. Nishimura[*], Yoshihiro Tamai[†]

July 2017

Abstract

This paper investigates the long-run welfare implication of monetary policy of a central bank in aging economy, especially, the effects of monetary policy on the prices of real properties. Helicopter money in the population bonus period enables young people to transfer purchasing power from the present to a distant future. Comparing to the economy where nominal stock of money is fixed regardless of the demographic transition, helicopter money for price stability suppresses extreme soar of the prices of real properties and production capital and improves welfare.

Keywords: helicopter money, aging, overlapping generations

JEL Classification: E58, G11, G12, J11

[*]Graduate School of Economics, University of Tokyo, 7-3-1, Hongo, Bunkyo-Ku, Tokyo 113-0033 Japan, and National Graduate Institute for Public Policy, 7-22-1 Roppongi, Minato-ku, Tokyo 106-8677 Japan

[†]Department of Economics, Kanagawa University, 3-27-1 Rokkakubashi, Kanagawa-Ku, Yokohama-Shi, Kanagawa 221-8686 Japan

1. Introduction

Many economies in the world will soon be or have already been aging rapidly. Figure 1 depicts the pace of aging in selected developed and emerging economies, which is based on United Nations' Population Prospects. Japan is a spearhead of the worldwide population aging[1], and other developed economies follow suit. Even some emerging economies will soon face the problem in their pursuit for economic development.

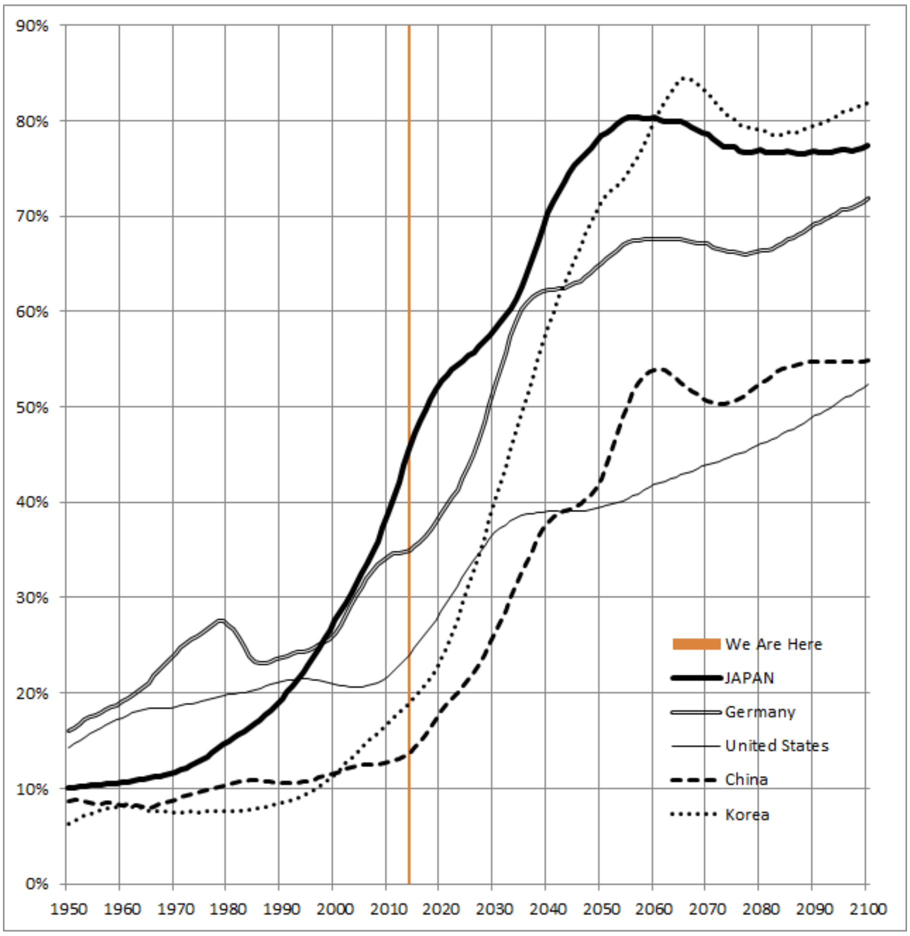
This paper investigates the inter relation between the central banks' monetary policy scheme and the asset price such as the price of the land for residence in such an astoundingly rapidly aging economy, and the welfare implication of the monetary policy scheme.

Many authors have been studied the optimality of monetary policy in relation to the people's portfolio choice between productive capital and money, and analyzed Tobin effect (Battacharya et. al. (2009) [3], Yakita (2006) [23]). Positive nominal inflation rate can be preferable promoting the demand for nonmoney asset and capital formation.

On the other hand, if the economy is subject to severe demographic transition from population bonus phase to population onus phase, the price of an asset whose stock is exogenously limited such as land (real property) varies rapidly as many empirical studies show [2] Thus, the demand for properties (their land components) and their prices are determined by people's choice of their very long run portfolio for retirement. In this very long run portfolio choice, there is another non-depreciable asset, which is money. Although money

[1] There are many studies about a substantial growth slowdown and its policy implications in recent years (see for example, Nishimura and Shirai 2003[15] and Nishimura and Saito 2004[14] in the case of Japan,). However, demographic factors have not been fully discussed in this context.

[2] Many researches have been devoted to study the effects of demographic factors on property prices. See Mankiw and Weil (1989)[11], DiPasquale and Wheaton (1994)[5], Engelhardt and Poterba (1991)[6], Hamilton (1991)[7], Hendershott (1991)[8], Kearl (1989)[10], and Poterba (1984)[18] for the United States, and Ohtake and Shintani (1996)[17] and Saita, Shimizu and Watanabe (2016)[19] for Japan. However, they are mostly base short-run demand and supply relations. Our analysis is based on very long run portfolio choice and in the same direction as Takáts (2012)[21] and Nishimura and Takáts (2012)[16] and Tamai, Shimizu and Nishimura (2017) [22].



Source: United Nations World Population Prospects, 2012 Revision.

Figure 1: Rapidly Aging Developed and Emerging Economies: Old-Age Dependency Ratio

was clearly rate-dominated by other assets and thus excluded from the very long run portfolio in the past high inflation eras, the recent price stability makes money become an important asset class for people to prepare for their retirement.

In fact, when the economy's inflation rate is high, holding a large amount of nominal money is not a wise strategy in asset management. This is especially so when people are considering very distant future, say, thirty years from now. Then, it is safe to assume away nominal money from a very long run portfolio and to postulate nominal money is held only for transaction purposes. In effect, money is a veil.

However, since the 1980s, so-called Great Moderation of tamed inflation has been achieved. Moreover, we have been witnessing dis-inflationary or even deflationary trend to date. This change has been brought by a change in the monetary policy regime, in which central banks now explicitly target price stability by inflation targeting. They now make it clear that price stability is their mandate, which does not change in the future.

The most important consequence, which is not understood well unfortunately, is that money becomes an important asset even in a very long run portfolio. A good example is Japan. For more than two decades of almost zero inflation, people are holding a large amount of money in the form of bank deposits. In fact, during this period, money as a very long run asset has fared well compared with stock markets' and property markets' performance.

In such an occasion, portfolio choice between money and land is critical for the welfare of an individual. With the transition from the population bonus to onus phase, monetary policy that is oriented to the quantitative stability of money makes inflation more serious, while the policy oriented to price stability (inflation targeting) makes those who were born in the population bonus period better off.

As for the relation between population aging and optimal monetary policy, Yakita (2006) [23] investigates two-period-lived-agent overlapping generations economy with portfolio choice between money and productive capital, and shows that population aging in terms of longevity (decreasing in mortality rate) increases real savings and enhances economic growth if growth rate of money is high enough to generate positive inflation rate.

In this paper, on the other hand, population aging is not represented by the mortality rate but is drastic change in the population size of generations, and portfolio choice between money and land is analyzed. Monetary policy oriented to price stability rather than inflation improves welfare mitigating fluctuation of land price and providing money balance as a good store of value for those in their working age.
The rest of this paper is organized as follows. In Section **??**, we develop a theory of very long run portfolio choice between these two non-depreciable assets: one is real (land) and the other is nominal (money), in an economy in transition from young and growing population to rapidly aging one. It is shown that aging has profound negative effects on (very long run) real property prices, and that the monetary regime is a key factor influencing (very long run) real property prices. In particular, real property prices in the population bonus phase are higher in a constant-monetary-quantity regime such as gold standards than in an inflation-targeting monetary regime.

In section 3 welfare implications of monetary policy is discussed.
In section 4 the possibility of extension of the model of section 2 is discussed. In addition to the land and money, productive capital is introduced. As for constant monetary quantity regime, inter dependence between demographic transition and change in prices is almost same as the results in section 2 even if we introduce productive capital. Section 5 provides some concluding remarks.

2. A Theory of Very Long Run Portfolio Choice

2.1 Model Setups

To setup a theory of very long run portfolio choice between two non-depreciable assets (land and money), we employ a stylized two-period-live-agent overlapping generations model, following Allais (1947)[1], Samuelson (1958)[20]. A young individual work for real wage income the value of which is exogenously given as y^Y in terms of consumption goods. Consumption goods is perishable within a period. Saving is done through a divisible utility-bearing real asset called land and through utility-bearing money. Old agents do not work; they sell their accumulated assets (land and money) and consume.

At time t, there are n_t young agents; hence, at time $t+1$ there are n_t old agents. Formally, individual agents' utility function (U) can be written as follows:

$$U\left[c_t^Y, c_{t+1}^O, h_t, M_t/P_t\right] \equiv \eta \ln(c_t^Y) + \mu(1-\eta)\ln\left(\frac{M_t}{P_t}\right) + (1-\mu)(1-\eta)\ln(h_t) + \beta \ln(c_{t+1}^O), \tag{1}$$

where $\ln(\cdot)$ is the natural logarithm, c^Y is consumption when young, and c^O is consumption when old. η and μ are parameters within the interval $(0,1)$. $0 < \beta < 1$ is the discount factor, and t is the time period index.

The budget constraint for a generation-t individual when he/she is young and old are

$$c_t^Y \leq y^Y - q_t h_t - M_t/P_t, \tag{2}$$

$$c_{t+1}^O \leq q_{t+1} h_t + M_t/P_{t+1}, \tag{3}$$

respectively, where P_t and P_{t+1} are the consumption goods price in period t, and $t+1$, q_t and q_{t+1} are the <u>real</u> price of the land in terms of the value of consumption goods in each period.

We examine a stylised demographic transition, which captures the phase transition from a demographic bonus phase to a demographic onus phase. Table 1 summarises the stages of this stylised demographic transition. The economy starts in a steady state ($t=0$) with population size at $n+\gamma$. Then, unexpectedly, the population increases to $n+\Delta$ ($t=1$, baby boom, where $0<\gamma<\Delta$). In the baby boom period, there are more young productive workers than old people, which can be thought of as a demographic bonus. However, the next generation is assumed to be smaller at size n ($t=2$, aging period), which implies that old people now outnumber the working-age population. In the following period, the system stabilises at this new, lower population steady state ($t=3,4,\ldots$).

2.2 Demand for Land and for Real Money Holdings

The demand for land and for real money holdings of each generation-t young are determined by his/her own life-time utility maximization. By the linear

Table 1: Demographic Transition

Time	Young Population Size	Old Population Size	Name of Period
$t = 0$	$n + \gamma$	$n + \gamma$	old steady state
$t = 1$	$n + \Delta$	$n + \gamma$	baby boom
$t = 2$	n	$n + \Delta$	aging
$t = 3, 4, \ldots$	n	n	new steady state

homogeneity of Cobb-Douglas utility function like (1), the utility maximization can be decomposed into two phases: (i) optimal allocation of the endowment y^Y into consumption and saving when he/she is young (consumption and saving choice), and (ii) how the saving should be divided into land and real money (portfolio choice).

The first order conditions for the maximization of generation-t lifetime utility subject to budget constraints (2) and (3) are as follows.

$$\frac{-\eta q_t}{y^Y - h_t q_t - M_t/P_t} + \frac{(1-\mu)(1-\eta)}{h_t} + \frac{\beta q_{t+1}}{h_t q_{t+1} + M_t/P_t} = 0, \tag{4}$$

$$\frac{-\eta/P_t}{y^Y - h_t q_t - M_t/P_t} + \frac{\mu(1-\eta)}{M_t} + \frac{\beta/P_{t+1}}{h_t q_{t+1} + M_t/P_t} = 0 \tag{5}$$

$(4)\times h_t + (5)\times M_t$ implies that the optimal saving rate $s_t^* \equiv (q_t h_t + M_t/P_t)/y^Y$ is constant at $s^* = 1 - \frac{\eta}{1+\beta}$ regardless of real land prices q_t and q_{t+1} and goods and services prices P_t and P_{t+1}.

Thus, land prices and consumption goods prices influence the economy only through portfolio choice between land and real money balances. By the property of the Cobb-Douglas utility function, the optimal ratio of land to real money is determined by relative rate of return of land with respect to real money. For notational simplicity, we use notations shown by Table 2 in the following discussion.

$(4) \times h_t - (5) \times P_t \times h_t \times q_t$ yields

$$(1-\mu)(1-\eta) + \frac{\beta \rho_t \theta_t}{1 + \rho_t \theta_t} = \mu(1-\eta)\theta_t + \frac{\beta \theta_t}{1 + \rho_t \theta_t}, \tag{6}$$

which implies that $\theta_t > 0$ is a strictly increasing function of ρ_t, $\theta_t = \theta(\rho_t)$. It is obvious by (6) that $\theta(1) = (1-\mu)/\mu$, that is, if land is equivalent to the

Table 2: Notations

$s_t^* \equiv (h_t q_t + M_t/P_t)/y^Y$	Optimal saving rate : $s_t^* = s^* \equiv 1 - \frac{\eta}{1+\beta}$ (constant)
$\theta_t \equiv q_t h_t/(M_t/P_t)$	Ratio of land investment to real money investment
q_{t+1}/q_t	Rate of return on land for the period-t young
P_t/P_{t+1}	Rate of return on real money for the period-t young
$\rho_t \equiv (q_{t+1}/q_t) \div (P_t/P_{t+1})$	Relative rate of return of land to real money
$z_t \equiv P_t q_t$	Nominal price of land

money holdings with respect to the rate of return, it is optimal for the young to divide his/her savings between land and real money by ratio of $1-\mu : \mu$. It can also be verifiable that $\theta'(\cdot) > 0$ (see the Appendix A). The demand for the real land h_t^d and for the money M_t^d are determined by the fact that the optimal saving rate is s^* and the optimal portfolio choice $(q_t^d h_t)/(M_t^d/P_t)$ equals to $\theta(\rho_t)$. In aggregate forms, we have

$$n_t h_t^d z_t + n_t M_t^d = n_t s^* y^Y P_t \tag{7}$$

and

$$\frac{n_t h_t^d z_t}{n_t M_t^d} = \theta(\rho_t). \tag{8}$$

2.3 Supply of Assets

We assume that the aggregate supply of land is exogenously given and constant at H^*.

As for the aggregate money supply, we will discuss two regimes, 1) Constant Monetary-Quantity Regime (Quantity Stability), and 2) Inflation-Targeting Regime (Price Stability). In the former, the aggregate money supply is exogenously constant at M^*, as is the case of the gold-standard regime. In the latter case, each of the young demands money holdings under the expectation that the price is constant at some level $\overline{P}$, while the central bank supplies money as much as the quantity that is consistent with the young's expectation. We will consider the equilibrium in each of these two regimes in the subsequent subsections.

It should be noted here that this overlapping-generation economy is inherently dynamic, and prices are so-called jumping variables. To determine the equilibrium path of the economy, we assume that the economy should not explode, or equivalently, that the economy should converge to a steady state.

2.4 Constant Monetary Quantity (CMQ) Regime (Quantity Stability)

The constant monetary quantity regime is quite similar to a strict gold standard regime: even though paper money exists, it behaves as if it is fully backed by gold.

Formally, we set aggregate money supply at constant M^* in the model. In this regime, the ratio of real land to the money holding is constant and the equilibrium condition implies

$$\frac{H^* z_t}{M^*}\left(=\frac{n_t h_t^d z_t}{n_t M_t^d}\right)=\theta\left(\frac{z_{t+1}}{z_t}\right) \tag{9}$$

The equation (9), which is the difference equation of z_t and z_{t+1} , determines the dynamics of the nominal price of land. In fact, (9) implies that, by setting $z_t = z_{t+1} = z^*$, $z^* \equiv M^*/H^*$ is the unique steady state. Moreover, this steady state is locally unstable (see the Appendix B), hence the immediate jump to the steady state is the only path that converge to the steady state. Therefore, $\theta_t = \frac{1-\mu}{\mu} \quad \forall t$ and the aggregate savings of the young is divided between land and real money balances by the ratio of $1-\mu : \mu$ every period, that is, $H^* q_t = (1-\mu) n_t s^* y^Y$ and $M^*/P_t = \mu n_t s^* y^Y$.
Thus, both the real value of land and the real value of money (*i.e.*, the inverse of the price level) are proportionate to the aggregate real savings, which is proportionate to aggregate real income of the young, $n_t y^Y$. Hence the real price of land and the price of goods and services depend on both economic and demographic factors as follows.

$$q_t = \frac{n_t}{H^*}(1-\mu)s^* y^Y \quad \text{and} \quad P_t = \frac{M^*}{n_t}\frac{1}{\mu s^* y^Y}. \tag{10}$$

2.5 Inflation Target (IT) Regime (Price Stability)

Under inflation targeting, we consider a fully elastic money supply. We assume that there is an inflation targeting central bank that stabilises the price level at $\overline{P}$. The central bank supplies money to keep the price level constant.

In the overlapping generation framework, how money is supplied matters. We assume the following helicopter-drop procedure. At the dawn of period t, the old generation has aggregate money holdings $n_{t-1}M_{t-1}$. When this amount is not equal to the amount n_tM_t necessary to keep the price level constant, the central bank dispatches helicopter squads to drop the difference to the old generation's home before the markets open. (If the difference is negative, helicopter squads seize the difference from the old generation.) When the markets open, the monetary stocks of the old generation are n_tM_t, which is demanded by the young generation. The helicopter drop (or seizure) is assumed to be unexpected for the old generation when they are young in the previous period.

In general, combining (7) and (8) and that *per capita* money supply for the young M_t equals to money demanded by each agent M_t^d, $H^*z_t(1+1/\theta(\rho_t)) = n_tP_ts^*y^Y$ and $H^*z_{t+1}(1+1/\theta(\rho_{t+1})) = n_{t+1}P_{t+1}s^*y^Y$ hold. Dividing the former by the latter side by side,

$$\frac{z_t}{z_{t+1}}\left[\frac{1+1/\theta(\rho_t)}{1+1/\theta(\rho_{t+1})}\right] = \frac{n_tP_t}{n_{t+1}P_{t+1}}. \tag{11}$$

Inflation targeting can be written formally as $P_t = \overline{P}$ for all t.[3]

As customary in such a dynamic framework, we will solve this difference equation backward from the future.

2.5.1 Equilibrium from the period 2 and thereafter

[3]It is evident that the inflation target of x% can easily be incorporated in this difference equation.

Under the demographic transition shown by Table 1, the population size of the younger generation is constant for all $t \geq 2$, hence (11) implies

$$\frac{1 + 1/\theta(\rho_t)}{1 + 1/\theta(\rho_{t+1})} = \rho_t \quad \text{for all } t \geq 2. \tag{12}$$

Since $\theta'(\cdot) > 0$, (12) implies $1 \lesseqgtr \rho_t \lesseqgtr \rho_{t+1} \lesseqgtr \rho_{t+2} \lesseqgtr \cdots$, hence the only path of z that converges to the steady state is the immediate jump of z_2 to the steady state. Let M^{SS}, z^{SS}, and $q^{SS} \equiv z^{SS}/\overline{P}$ denote the steady state values of *per capita* demand for nominal money of the young, nominal and real value of the real land price respectively. (7) and (8) imply

$$\frac{M^{SS}}{\overline{P}} = \mu s^* y^Y, \quad \text{and} \quad q^{SS} = \frac{n}{H^*}(1-\mu) s^* y^Y. \tag{13}$$

It is noteworthy that q^{SS} is the same as the real land price in the constant monetary quantity regime from the period 2 and thereafter.

2.5.2 Equilibrium of the period 1 (population bonus period)

In period 1, the population of the young is $n + \Delta (> n + \gamma)$. This population bonus is unexpected for the old generation, while we assume that the young generation in the period 1 expect that the population size of each generation from the period 2 and on will be n, and that the equilibrium will be the steady state as shown by (13).

Let us first give an intuitive interpretation of the difference between the inflation target regime and constant monetary quantity regime.

In the population bonus period, young people rationally expect that the population will shrink next period. This shrink causes inflation in the next period (period 2) in QMS regime, while in the IT regime, no inflation is expected. This implies that the rate of return on land in the inflation target regime is smaller than that in the constant monetary quantity regime for the period 1 young generation, so that the real land price is lower in the inflation target regime than in the constant monetary quantity regime.

Formally, in the inflation target regime, (11) as for $t = 1$ implies

$$(1-\mu)\left\{1 + \frac{1}{\theta[(q^{SS}/q_1) \div (P_1/P_2)]}\right\} = \frac{n+\Delta}{n}\frac{q^{SS}}{q_1}. \tag{14}$$

Given the rate of return on real money holdings for the young in the period 1, (P_1/P_2), (14) gives the equilibrium rate of return of the real land (q^{SS}/q_1). Since θ, rate of expenditure on the real land to the expenditure on the real money holdings, is increasing in (q^{SS}/q_1) and decreasing in (P_1/P_2), the left-hand side is decreasing in (q^{SS}/q_1) and increasing in (P_1/P_2). Right-hand side is obviously increasing in (q^{SS}/q_1). Therefore, the larger is the rate of return on money (P_1/P_2), the larger is the equilibrium (q^{SS}/q_1). As (10) shows, the price of goods and services is proportionate to the inverse of the population size of the young in each period in the constant monetary quantity regime, so the young generation in the period 1, expecting the decreasing of population of the next generation, expect the inflation when they would be old, and the rate of return of money (P_1/P_2) is $n/(n+\Delta) < 1$, while in the inflation target (price stability) regime on the other hand, P_1/P_2 is unity. Therefore, by (14), we obtain

$$\frac{n}{n+\Delta} = \frac{q^{SS}}{q_1^{CMQ}} < \frac{q^{SS}}{q_1^{IT}} < 1 \quad \text{or} \quad q_1^{CMQ} > q_1^{IT} \tag{15}$$

where q_1^{CMQ} and q_1^{IT} are, period-1 real price of the land in the constant monetary quantity regime and in the inflation targeting regime, respectively. As expected, the real land price is lower in the inflation target regime than the constant monetary quantity regime.

The following proposition shows that the rate of *per capita* money demand in period 2 to that in period 1, M_2/M_1 is less than unity. Comparing this with the fact that the rate of per capita nominal money supply in period 2 to that in period 1 $\frac{M^*/n}{M^*/(n+\Delta)} > 1$ in the quantity stability regime, this proposition implies that the central bank makes positive helicopter money drop in population bonus period under the IT regime.

Proposition 1 *In the inflation target regime, the rate of per capita money demand in period 2 to that in period 1, M_2/M_1 is less than unity.*
Proof. *See Appendix C.* ■

2.5.3 Equilibrium up to the period 0

Up to the period 0, the population of each generation is constant at $n + \gamma$, and (12), and thus (7) and (8) imply

$$\frac{M_0}{\overline{P}} = \mu s^* y^Y, \quad \text{and} \quad q_0 = \frac{n+\gamma}{H^*}(1-\mu)s^* y^Y \tag{16}$$

3. The Sequences of Land Price, Real Money Balances and Welfare

In this section we compare two monetary policy regimes discussed in the last section, quantity stability regime and inflation target regime, with respect to the equilibrium sequences of real land prices q_t and that of real money balances M_t/P_t to investigate the welfare implication of the helicopter drops and raid that is oriented to the stability of consumption goods price in the economy where people can make portfolio choices between money and money.

3.1 Real Land Prices and Real Money Balances in Demographic Transition

Table 3 and Table 4 summarize the transition of real land prices q_t and real money balances $\frac{M_t}{P_t}$ in demographic transition shown by (10), (13), (15), (16) and the proposition 1, where $q^* \equiv (1-\mu)s^* y^Y / H^*$.

It is useful to recognize that real land prices are the same under the constant monetary quantity and inflation target regimes in all periods, *except for the population bonus period* (period $t = 1$) as Table 3 summarizes. That is, monetary regimes are neutral except for the period of demographic transition. (15) shows that the real land prices are lower under inflation target than under fixed money supply at time $t = 1$.

Table 3: Real land prices in demographic transition

	$t = 0$	$t = 1$	$t = 2$	$t = 3$	$t = 4\ldots$
Constant money supply	$q^*(n+\gamma)$	$q^*(n+\Delta)$	$q^* n$	$q^* n$	$q^* n$
Inflation targeting	$q^*(n+\gamma)$	q_1	$q^* n$	$q^* n$	$q^* n$

Table 4: Real Money Balances in demographic transition

	$t=0$	$t=1$	$t=2$	$t=3$	$t=4\dots$
Constant money supply	$\mu s^* y^Y$	$\mu s^* y^Y$	$\mu s^* y^Y$	$\mu s^* y^Y$	$\mu s^* y^Y$
Inflation targeting	$\mu s^* y^Y$	$\mu s^* y^Y + \sigma$	$\mu s^* y^Y$	$\mu s^* y^Y$	$\mu s^* y^Y$

By (15), we have $q^*(n+\Delta) > q_1$, and by proposition 1, $\sigma > 0$.

In sum, Table 3 shows that (1) demographic factors are important determinants of land prices in the very long run, and (2) monetary policy regimes (or whether we have price stability or not) greatly influence real land prices of the demographic transition period.[4]

3.2 Welfare Implication of Helicopter Money Drops and Raid for Price Stability

The results shown by Table 3 and 4 contain welfare implication of the IT regime as follows.

The sources of utility of a generation-t individual are c_t^Y, c_{t+1}^O, M_t/P_t, and h_t. Since equilibrium real saving rate s^* is constant, c_t^Y is constant regardless of monetary policy regime. In equilibrium, aggregate consumption of the old ex post is the same as the aggregate savings of the young in every period, which is a constant share of the aggregate income of the young $n_t y^Y$, which is independent of the monetary policy regime.

Since the aggregate supply of the land is constant H^*, the sequence of h_t in equilibrium is H^*/n_t, which is independent of monetary policy regime.

Therefore, monetary policy regime affects welfare only through the equilibrium real money balances. Table 4 and proposition 1 imply that inflation target regime for price stability with helicopter money drops and raids dominates the quantity stability regime in terms of lifetime utility of the generation 1 (those who are born in the baby boom period), since the price stability makes the real money balances more valuable asset for generation-t

[4] Nishimura and Takáts (2012)[16] examined a panel of twenty-two advanced economies over the 1950-2011 period to empirically confirm the theory. They found that baby boomers' saving demand drove both property prices and money demand higher.

individuals.

4. Extension: a Model with Productive Capital

In the model discussed in section 2 and 3, the real income of the young is exogenous. In this section we discuss the possibility of the extension of the model towards overlapping generation economy with productive capital as a variation of Diamond (1965) [4], where people can hold productive capital as an asset in addition to holding the real money balances and the real properties (land).

4.1 Model Setups

4.1.1 Individuals

The utility function is the same as (1). At the beginning of each period, young people are born with an endowment of a unit of labor, which is supplied inelastically when they are young. The real wage w_t payed for the unit labor supplied by generation-t young worker is only the source of income for the individual during the young age. Generation-t young individual allocates the income of w_t into his/her own consumption c_t^Y and savings, which can be invested for three kinds of assets, i.e., productive capital (capital goods), real properties such as land for residence, real money balances.

4.1.2 Production Sector

The factors of production are capital and labor. Labor is supplied by the young and capital is supplied by the old. Capital goods purchased by generation-t young is put into production in the period $t+1$, and the per capita quantity of capital purchased by a generation-t individual in period t is denoted by k_{t+1}. Both of capital goods and consumption goods are produced by the same constant-returns-to-scale technology , and the marginal rate of transformation between capital goods and consumption goods is unity. Both of the production sectors of capital goods and consumption goods are perfectly

competitive, hence the real price of capital goods measured by consumption goods is unity.
We also assume that all of the productive capital depreciates once it is put into the production process, that is, we assume that the depreciation rate of the productive capital is 100%.
Therefore, the equilibrium of the flow of capital goods and consumption goods in period t is

$$n_t c_t^Y + n_{t-1} c_{t-1}^O + n_t k_{t+1} = F(n_{t-1} k_t, n_t)$$

where c_t^Y and c_{t-1}^O are period-t consumption of the young and the old per head, $n_t k_{t+1}$ is the young's aggregate investment for the capital goods that will be input for the production in the period $t+1$ (when period-t young will be old), and $F(\cdot, \cdot)$ is the production function that represents constant returns to scale technology. The production function is specified by the Cobb=Douglas one such as $F(K, L) \equiv K^{1-\alpha} L^{\alpha}$.

4.2 Demand for Consumption Goods and Assets

4.2.1 Budget Constraints

By the setups in the section 4.1, the budget constraints of a generation-t individual at two stages of his/her life are

$$c_t^Y = w_t - k_{t+1} - \frac{M_t}{P_t} - q_t h_t \tag{17}$$

$$c_{t+1}^O = (1 + r_{t+1}) k_{t+1} + \frac{M_t}{P_{t+1}} + q_{t+1} h_t \tag{18}$$

where r is the real rate of return (rental price) for the the production capital.

4.2.2 Demand for Consumption Goods, Capital Goods, Real Money Balances and Real Property

The first order conditions for the maximization of the utility shown by (1) subject to budget constraints (17) and (18) are

$$\frac{-\eta}{c_t^y} + \beta \cdot \frac{1 + r_{t+1}}{c_{t+1}^o} = 0 \tag{19}$$

$$\frac{-\eta/P_t}{c_t^y} + \mu(1-\eta) \cdot \frac{1}{M_t^d} + \beta \cdot \frac{1/P_{t+1}}{c_{t+1}^o} = 0 \tag{20}$$

$$\frac{-\eta q_t}{c_t^y} + (1-\mu)(1-\eta)\frac{1}{h_t^d} + \frac{\beta q_{t+1}}{c_{t+1}^o} = 0 \tag{21}$$

Demand for Consumption Goods Considering (19) $\times k_{t+1}$+(20) $\times M_t^d$+ (21) $\times h_t^d$ and (17), we obtain demand for consumption goods of period-t young and period-$(t+1)$ old as

$$c_t^Y = \frac{\eta}{1+\beta} w_t \tag{22}$$

$$c_{t+1}^O = \frac{\beta}{\eta}(1 + r_{t+1}) c_t^y = \frac{\beta}{1+\beta}(1 + r_{t+1}) w_t. \tag{23}$$

Demand for Assets We can obtain demand for real money balances by substituting (22) and (23) into (20) and rearranging. It is noteworthy that demand for real money balance is positive only if $\frac{P_t}{P_{t+1}} \frac{1}{1+r_{t+1}} < 1$, which is equivalent to the condition that nominal interest rate is positive, as Yakita (2006) points out.

Substituting (22) and (23) into (21) and rearranging yields the expenditure for real property of each young individual.

Considering these, individual demand for real money balances, expenditure on land and on productive capital are as follows.

$$\frac{M_t^d}{P_t} = \frac{\mu(1-\eta)}{1+\beta}\frac{1}{1-\frac{P_t}{P_{t+1}}\frac{1}{1+r_{t+1}}}w_t \tag{24}$$

$$q_t h_t^d = \frac{(1-\mu)(1-\eta)}{1+\beta}\frac{1}{1-\frac{q_{t+1}}{q_t}\frac{1}{1+r_{t+1}}}w_t \tag{25}$$

$$\begin{aligned} k_{t+1} &= w_t - c_t^y - \frac{M_t^d}{P_t} - q_t h_t^d \\ &= \left[\left(1-\frac{\eta}{1+\beta}\right) - \frac{\mu(1-\eta)}{1+\beta}\frac{1}{1-\frac{P_t}{P_{t+1}}\frac{1}{1+r_{t+1}}}\right. \\ &\quad \left. - \frac{(1-\mu)(1-\eta)}{1+\beta}\frac{1}{1-\frac{q_{t+1}}{q_t}\frac{1}{1+r_{t+1}}}\right] w_t \end{aligned} \tag{26}$$

4.3 Dynamics and Equilibrium for the Quantity Stability Regime

By the assumption of perfect competition and the 100% of capital depreciation, we can conclude that per capita wage rate equals to the marginal productivity of labor and that $1 + r_{t+1}$ coincides with the marginal productivity of capital in period $(t + 1)$. To be specific,

$$w_t = \alpha\left(\frac{n_{t-1}k_t}{n_t}\right)^{1-\alpha} \tag{27}$$

which implies $n_t w_t = \alpha(n_{t-1}k_t)^{1-\alpha}n_t^{\alpha}$, the share of aggregate labor income is α. and

$$1 + r_{t+1} = (1-\alpha)\left(\frac{n_t k_{t+1}}{n_{t+1}}\right)^{-\alpha} \tag{28}$$

4.3.1 Equilibrium Path of q, M, and k: Constant Monetary Quantity Regime

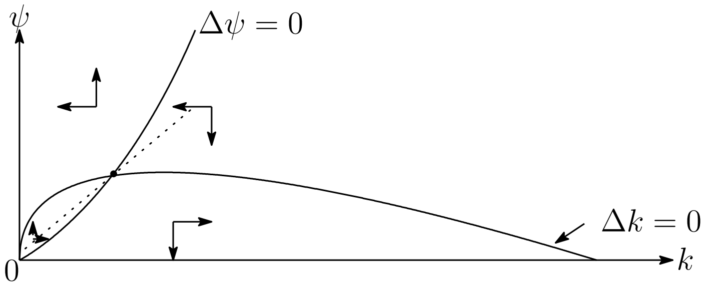
Substituting (27) and (28) into individual demand for assets, (24), (25) and (26) and considering aggregate equilibrium in asset markets, we obtain basic system of difference equations for the dynamic path of q, M, and k. As for Constant Monetary Quantity Regime, the system can be reduced to the following two equations (for derivation, see Appendix D).

$$n_t k_{t+1} = \left(1 - \frac{\eta}{1+\beta}\right) \alpha (n_{t-1} k_t)^{1-\alpha} n_t^{\alpha} - \psi_t \tag{29}$$

$$\beta \psi_t - (1+\beta-\eta)\psi_{t+1} \frac{1}{1-\alpha} \left(\frac{n_t k_{t+1}}{n_{t+1}}\right)^{\alpha} = (1-\eta) n_t k_{t+1} \tag{30}$$

where $\psi_t \equiv \frac{M^*}{P_t} + q_t H^*$, aggregate expenditure on real money balances and real properties in equilibrium of the CMQ regime.

If population size of each generation is constant, difference equations (29) and (30) imply phase diagram as for k and ψ as follows. If constant population continues for a long time, only the path that is consistent with the long-run optimality is the saddle point path that converges to the steady state.



4.3.2 Steady State, Equilibrium Portfolio and the Path of k, P and q

Steady State Level of Capital Investment The steady state capital investment per capita k that (29) and (30) imply for a constant size of population is the solution of the following equation.

$$\frac{\alpha(1-\eta)}{1+\beta} = \left(1 - \frac{1}{1-\alpha}\right) \left[\alpha \left(1 - \frac{\eta}{1+\beta}\right) - k^{\alpha}\right]. \tag{31}$$

(31) is a quadratic equation with respect to k^{α}. Since $\frac{1-\eta}{1+\beta} < 1 - \frac{\eta}{1+\beta}$, one of the real root of the quadratic equation satisfies $0 < k^{\alpha} < (1-\alpha)$. This root determines the per capita productive capital in the steady state and let $\overline{k}$ denote this value.

Steady State Equilibrium up to Period 0 Suppose that the k_t has long been this steady state $\overline{k}$ for a long time up to period 0. Let P_0 and q_0 be consumption goods price and land price up to period 0 respectively. Considering (24), (25) at steady state, (27), (28) and the equilibrium conditions of real money balance and land in the CMQ regime ($(n+\gamma)M^d/P_0 = M^*/P_0$, $(n+\gamma)h^d = H^*$), we obtain

$$P_0 = \frac{M^*}{n+\gamma}\left[\frac{\mu(1-\eta)}{1+\beta}\overline{\Omega}\right]^{-1}, \quad q_0 = \frac{n+\gamma}{H^*}\frac{(1-\mu)(1-\eta)}{1+\beta}\overline{\Omega} \tag{32}$$

where $\overline{\Omega} \equiv \alpha\overline{k}^{1-\alpha}/\left[1-\overline{k}^{\alpha}/(1-\alpha)\right] = \alpha\overline{k}^{1-\alpha}/\left[1-1/(1+r)\right]$, discounted present value of per capita wage income of the young if population is expected to be constant.

Equilibrium Path of k, P, and q from Period 1 and Thereafter To obtain the path of P, q, and k thereafter, it is noteworthy that the individuals of any generation, in CMQ regime, make the same portfolio choice regardless of the population size of the generation to which they belong, as the following Lemma 1 and 2 show.

Lemma 2 *In the CMQ regime, ratio of aggregate expenditure on real money balances to aggregate expenditure on real properties is constant, to be specific,* $\frac{M^*/P_t}{q_t H^*} = \frac{\mu}{1-\mu}$ *holds for all t.*
Proof. *See Appendix E* ■

Lemma 3 *In the CMQ regime,* $x_t \equiv \frac{\psi_t}{n_t k_{t+1}}\left(= \frac{M^*/P_t + q_t H^*}{n_t k_{t+1}}\right)$ *is constant for all t.*
Proof. *See Appendix E* ■

Lemma 2 implies that $\frac{P_0}{P_t} = \frac{q_t}{q_0} = \frac{\psi_t}{\overline{\psi}}$, so that the paths of P and q are represented by the path of ψ. Applying Lemma 2 and 3 to the difference equation (29), we obtain the paths of k and ψ as shown by the following table (as for derivation, see Appendix F).
The Table 5 implies that $\overline{\psi} < \psi_1 > \psi_2 > \psi_3 > \cdots \to \frac{n}{n+\gamma}\overline{\psi}$. By Lemma 2, land price is proportionate to ψ and the price of consumption goods is

Table 5: Per capita Capital and Prices in demographic transition

		population bonus	population shrink	Convergence to the New steady state
	$t \leq 0$	$t = 1$	$t = 2$	$t \geq 3$
old	$n+\gamma$	$n+\gamma$	$n+\Delta$	n
young	$n+\gamma$	$n+\Delta$	n	n
$\frac{k_{t+1}}{k}$	1	$\left(\frac{n+\gamma}{n+\Delta}\right)^{1-\alpha}$	$\left(\frac{n+\gamma}{n+\Delta}\right)^{(1-\alpha)^2} \left(\frac{n+\Delta}{n}\right)^{(1-\alpha)}$	$\frac{k_{t+1}}{k} = \left(\frac{k_t}{k}\right)^{1-\alpha} \to 1$
$\frac{\psi_t}{\psi}$	1	$\left(\frac{n+\Delta}{n+\gamma}\right)^{\alpha}$	$\left(\frac{n+\gamma}{n+\Delta}\right)^{\alpha^2} \left(\frac{n}{n+\gamma}\right)^{\alpha} \frac{\psi_1}{\psi}$	$\frac{\psi_t}{\psi} = \left(\frac{\psi_{t-1}}{\psi_{t-2}}\right)^{1-\alpha} \frac{\psi_{t-2}}{\psi} \to \frac{n}{n+\gamma}$

inversely proportionate to ψ, the sequence of ψ_t implies inflation of land price (deflation of consumption goods price) in population bonus period, and down turn of land price (inflation of consumption goods price) of population shrink period and thereafter. The relation between prices and demographic transition in the economy with productive capital is quite similar to the equilibrium discussed in section 2 and 3.

5. Concluding Remarks

We have developed a theory of very long run portfolio choice between two non-depreciable assets of which one is real (land) and the other nominal (money) in non-inflationary environment, for an economy in transition from young and growing to rapidly aging population. Aging has been shown to have profound effects on real property prices, and that the monetary regime is a key factor influencing (very long run) real property prices.

In the demographic transition with rapid downturn of the population of working age, the monetary policy oriented for the quantitative stability may cause severe inflation in the course of population shrink. In the policy scheme where the central bank makes helicopter money drops and raid that is consistent with the people's expectation of the price stability, individuals are better off since the real money balance is more valuable asset for the young to invest.

There are limitations about the theory and empirical methods of this paper, which future research should address to. Firstly, only two generations in one point of time is also a restrictive assumption. Moreover, the way money is supplied in the model and people's expectations about it are also one specification among many possibilities. The incorporation of these features is the subject of future research.

References

[1] Allais, Maurice. 1947. *Economie et Intérêt.* Paris: Imprimierie Nationale

[2] Buttiglione, Luigi , Philip R. Lane, Lucrezia Reichlin and Vincent Reinhart. 2014. *Deleveraging? What deleveraging? Geneva Report on the World Economy* 16. London: Centre for Economic Policy Research

[3] Bhattacharya, Joydeep, Joseph Haslag and Antoine Martin. 2009. Optimal monetary policy and economic growth. *European Economic Review* 53 210–221.

[4] Diamond, Peter A. 1965. National debt in a neoclassical growth model. *American Economic Review*, 55 (5): 1126–50.

[5] DiPasquale, Denise, and William C. Wheaton. 1994. Housing market dynamics and the future of housing prices. *Journal of Urban Economics* 35 (1): 1–27.

[6] Engelhardt, Gary. V., and James M. Poterba. 1991. House prices and demographic change: Canadian evidence. *Regional Science and Urban Economics* 21 (4): 539–46.

[7] Hamilton, Bruce W. 1991. The baby boom, the baby bust, and the housing market: A second look. *Regional Science and Urban Economics* 21: 547–52.

[8] Hendershott, Patric H. 1991. Are real house prices likely to decline by 47 percent? *Regional Science and Urban Economics* 21: 553–63.

[9] Inoue, Tomoo, Chihiro Shimizu, Kiyohiko G. Nishimura. and Yongheng Deng. 2017. Aging, inflation and property prices, mimeo., University of Tokyo.

[10] Kearl, James R. 1989. Inflation, mortgages, and housing. *Journal of Political Economy* 87 (5): 1115–38.

[11] Mankiw, N. Gregory, and David N. Weil. 1989. The baby boom, the baby bust, and the housing market. *Regional Science and Urban Economics* 19 (2): 235–58.

[12] Nishimura, Kiyohiko. G. 2016. Three "seismic shifts" in the global economy and the policy challenges they pose. *International Finance* 19 (2): 219-29

[13] Nishimura, Kiyohiko. G. 2011. Population ageing, macroeconomic crisis and policy challenges. Paper presented at The Panel on The General Theory and the Policy Reponses to Macroeconomic Crisis, 75th Anniversary Conference of Keynes General Theory, 19–21 June, University of Cambridge, UK.

[14] Nishimura, Kiyohiko. G., and Makoto Saito. 2003. On alternatives to aggregate demand policy to revitalize the Japanese economy. *Asian Economic Papers* 2 (3), 87-126.

[15] Nishimura, Kiyohiko. G., and Masato Shirai. 2003. Can information and communication technology solve Japan's productivity-slowdown problem? *Asian Economic Papers* 2 (1), 85-136.

[16] Nishimura, Kiyohiko. G., and Előd Takáts. 2012. Ageing, property prices and money demand. BIS Working Papers, no. 385.

[17] Ohtake, Fumio., and Mototsugu Shintani. 1996. The effect of demographics on the Japanese housing market. *Regional Science and Urban Economics* 26 (2): 189–201.

[18] Poterba, James M. 1984. Tax subsidies to owner-occupied housing: an asset-market approach. *Quarterly Journal of Economics* 99 (4): 729–52.

[19] Saita, Yumi, and Chihiro Shimizu, and Tsutomu Watanabe. 2016. Aging and real estate prices: Evidence from Japanese and US regional data. *International Journal of Housing Markets and Analysis* 9 (1): 66-87

[20] Samuelson, Paul. 1958. An exact consumption-loan model of interest with or without the social contrivance of money, *Journal of Political Economy* 66 (6): 467–82.

[21] Takáts, Előd. 2012. Aging and house prices. *Journal of Housing Economics* 21 (2): 131–41.

[22] Tamai, Yoshihiro. Chihiro Shimizu, and Kiyohiko G. Nishimura. 2017. Aging and Property Prices: A Theory of Very Long Run Portfolio Choice and Its Predictions on Japanese Municipalities in the 2040s. *Asian Economic Papers* Forthcoming.

[23] Yakita, Akira. 2006. Life expectancy, money, and growth *Journal of Population Economics*, 19 (3): 579–592

Appendix A. The Properties of the Function $\theta(\rho_t)$

The equation (6) implies that θ_t is the positive root for the quadratic equation $f(\theta_t) \equiv \mu(1-\eta)\rho_t\theta_t^2 - \{\beta(\rho_t-1) + (1-\eta)[(1-\mu)\rho_t - \mu]\}\theta_t - (1-\mu)(1-\eta) = 0$. By differentiating the equation $f(\theta_t) = 0$, $[2\mu(1-\eta)\rho_t\theta_t - \{\beta(\rho_t-1) + (1-\eta)[(1-\mu)\rho_t - \mu]\}]d\theta_t + \theta_t\{\mu(1-\eta)\theta_t - [\beta + (1-\eta)(1-\mu)]\}d\rho_t = 0$. Evaluating at $f(\theta_t) = 0$, $[2\mu(1-\eta)\rho_t\theta_t - \{\beta(\rho_t-1) + (1-\eta)[(1-\mu)\rho_t - \mu]\}] = (1-\mu)(1-\eta)/\theta_t + \mu(1-\eta)\rho_t\theta_t$ hence

$$\frac{d\theta_t}{d\rho_t} = \theta_t \left\{ \frac{\beta + (1-\eta)(1-\mu) - \mu(1-\eta)\theta_t}{\frac{(1-\eta)(1-\mu)}{\theta_t} + \mu(1-\eta)\rho_t\theta_t} \right\} > 0 \tag{A1}$$

where the inequality holds since $\theta_t < \frac{\beta+(1-\eta)(1-\mu)}{\mu(1-\eta)}$ because $f\left(\frac{\beta+(1-\eta)(1-\mu)}{\mu(1-\eta)}\right) = \frac{\beta(1-\eta+\beta)}{\mu(1-\eta)} > 0$.

Appendix B. The Instability of the Steady State of (9)

Difference equation (9) implies

$$\frac{H^* z_{t+1}}{M^*} = \theta\left(\frac{z_{t+2}}{z_{t+1}}\right). \tag{B1}$$

Dividing (B1) by (9) side by side implies

$$\rho_t \theta(\rho_t) = \theta(\rho_{t+1}). \tag{B2}$$

Since $\theta(\cdot)$ is positive and strictly increasing, (B2) implies that

$$1 \lesseqgtr \rho_t \Rightarrow 1 \lesseqgtr \rho_{t+s} \quad \forall s \geq 1$$

by induction, which is equivalent to $z_t \lesseqgtr z_{t+1} \lesseqgtr z_{t+2} \lesseqgtr \cdots$, hence the steady state of (9) is locally unstable.

Appendix C. Proof of Proposition 1

In equilibrium, the ratio of aggregate nominal investment on land ($H^* z_t$) to the aggregate demand for the nominal money holdings ($n_t M_t$) should be equal to $\theta(z_{t+1}/z_t)$, that is

$$\frac{H^* z_t}{n_t M_t} = \theta\left(\frac{z_{t+1}}{z_t}\right) \tag{C1}$$

and

$$\frac{H^* z_{t+1}}{n_{t+1} M_{t+1}} = \theta\left(\frac{z_{t+2}}{z_{t+1}}\right). \tag{C2}$$

Dividing (C1) and (C2) side by side,

$$\frac{z_t}{z_{t+1}} \frac{n_{t+1} M_{t+1}}{n_t M_t} = \frac{\theta\left(\frac{z_{t+1}}{z_t}\right)}{\theta\left(\frac{z_{t+2}}{z_{t+1}}\right)}. \tag{C3}$$

If $t = 1$, (C3) is equivalent to

$$\delta \equiv \frac{n_2 M_2}{n_1 M_1} = \frac{\mu}{1-\mu} \theta\left(\frac{z_2}{z_1}\right) \frac{z_2}{z_1} = \frac{\mu}{1-\mu} \theta(\rho_1) \rho_1 \tag{C4}$$

where $\delta \equiv (n_2 M_2)/(n_1 M_1)$ is expansion rate of aggregate money.
On the other hand, (14) implies

$$1 + \frac{1}{\theta(\rho_1)} = \frac{1}{1-\mu} \cdot \frac{n+\Delta}{n} \frac{P_1}{P_2} \frac{z_2}{z_1}. \tag{C5}$$

Multiplying both sides by $\theta(\rho_1)$,

$$\theta\left(\frac{z_2}{z_1}\right) + 1 = \frac{1}{1-\mu} \cdot \frac{n+\Delta}{n} \frac{P_1}{P_2} \theta\left(\frac{z_2}{z_1}\right) \frac{z_2}{z_1}. \tag{C6}$$

Substituting (C4) into (C6) and rearranging, and considering that $P_1/P_2 = 1$,

$$\theta(\rho_t) = \frac{n_1}{n_2} \frac{\delta}{\mu} - 1 \tag{C7}$$

Substituting (C4) and (C7) into (6), equation (6) is equivalent to

$$(1-\mu)(1-\eta) + \frac{\beta \frac{1-\mu}{\mu} \delta}{1 + \frac{1-\mu}{\mu}\delta} = \left(\mu(1-\eta) + \frac{\beta}{1 + \frac{1-\mu}{\mu}\delta}\right) \left(\frac{n+\Delta}{n} \frac{\delta}{\mu} - 1\right)$$

which implies the quadratic equation,

$$\begin{aligned} g(\delta) \equiv\, & \frac{(1-\eta)(1-\mu)}{\mu} \left(\frac{n+\Delta}{n}\right) \delta^2 \\ & + \left[\left(1 - \eta + \frac{\beta}{\mu}\right) \frac{n+\Delta}{n} - \frac{1-\mu}{\mu}(1+\beta-\eta)\right] \delta \\ & - (1+\beta-\eta) = 0 \end{aligned} \tag{C8}$$

By the sign of coefficients, the equation (C8) has a unique positive real root which is less than $\frac{n}{n+\Delta}$ since $g\left(\frac{n}{n+\Delta}\right) = \frac{1-\mu}{\mu}\beta \cdot \frac{\Delta}{n+\Delta} > 0$.
Therefore, $\mu < n/(n+\Delta) = n_2/n_1$ or equivalently,

$$\mu \equiv \frac{n_2}{n_1} \frac{M_2}{M_1} < \frac{n_2}{n_1} \quad \Leftrightarrow \quad \frac{M_2}{M_1} < 1 \tag{C9}$$

Appendix D Derivation of (29) and (30)

In CMQ regime, the aggregate quantities of nominal money stock and land are fixed at M^* and H^* respectively, so that equilibrium of asset markets are as follows.

$$\frac{M^*}{P_t} = \frac{n_t M_t^d}{P_t}, \quad q_t H^* = n_t q_t h_t^d \tag{D1}$$

Multiplying (24),(25), (26) by n_t and (27),(28), and considering (D1),

$$\frac{M^*}{P_t} = \frac{\mu(1-\eta)}{1+\beta} \frac{\alpha(n_{t-1}k_t)^{1-\alpha} n_t^\alpha}{1 - \frac{P_t}{P_{t+1}} \frac{1}{1-\alpha} \left(\frac{n_t k_{t+1}}{n_{t+1}} \right)^\alpha} \tag{D2}$$

$$q_t H^* = \frac{(1-\mu)(1-\eta)}{1+\beta} \frac{\alpha(n_{t-1}k_t)^{1-\alpha} n_t^\alpha}{1 - \frac{q_{t+1}}{q_t} \frac{1}{1-\alpha} \left(\frac{n_t k_{t+1}}{n_{t+1}} \right)^\alpha} \tag{D3}$$

$$n_t k_{t+1} = \left(1 - \frac{\eta}{1+\beta}\right) \alpha(n_{t-1}k_t)^{1-\alpha} n_t^\alpha - \frac{M^*}{P_t} - q_t H^* \tag{D4}$$

(D4) is the same as (29) since $\psi_t \equiv M^*/P_t + q_t H^*$.
Multiplying (D2) by $1 - \frac{P_t}{P_{t+1}} \frac{1}{1-\alpha} \left(\frac{n_t k_{t+1}}{n_{t+1}} \right)^\alpha$ and multiplying (D3) by $1 - \frac{q_{t+1}}{q_t} \frac{1}{1-\alpha} \left(\frac{n_t k_{t+1}}{n_{t+1}} \right)^\alpha$,

$$\frac{M^*}{P_t} - \frac{1}{1-\alpha} \left(\frac{n_t k_{t+1}}{n_{t+1}} \right)^\alpha \frac{M^*}{P_{t+1}} = \frac{\mu(1-\eta)}{1+\beta} \alpha(n_{t-1}k_t)^{1-\alpha} n_t^\alpha \tag{D5}$$

$$q_t H^* - \frac{1}{1-\alpha} \left(\frac{n_t k_{t+1}}{n_{t+1}} \right)^\alpha q_{t+1} H^* = \frac{(1-\mu)(1-\eta)}{1+\beta} \alpha(n_{t-1}k_t)^{1-\alpha} n_t^\alpha \tag{D6}$$

Summing (D5) and (D6) side by side,

$$\psi_t - \frac{1}{1-\alpha} \left(\frac{n_t k_{t+1}}{n_{t+1}} \right)^\alpha \psi_{t+1} = \frac{1-\eta}{1+\beta} \alpha(n_{t-1}k_t)^{1-\alpha} n_t^\alpha \tag{D7}$$

(29) implies $\alpha(n_{t-1}k_t)^{1-\alpha} n_t^\alpha = (n_t k_{t+1} + \psi_t) \left(1 - \frac{\eta}{1+\beta}\right)^{-1}$. Substituting this into the right-hand side of (D7) and rearranging yields (30).

Appendix E Proof of Lemma 2 and 3

Proof of Lemma 2

Rearranging (24) and (25) and multiplying by n_t,

$$\begin{aligned} \frac{M^*}{P_t} - \frac{1}{1+r_{t+1}} \frac{M^*}{P_{t+1}} &= \frac{\mu(1-\eta)}{1+\beta} n_t w_t \\ q_t H^* - \frac{1}{1+r_{t+1}} q_{t+1} H^* &= \frac{(1-\mu)(1-\eta)}{1+\beta} n_t w_t \end{aligned}$$

These equations imply that $\frac{M^*}{P_t} = \frac{\mu(1-\eta)}{1+\beta} \left(n_t w_t + \sum_{s=1}^\infty \prod_{\tau=1}^s \frac{1}{1+r_{t+\tau}} n_{t+\tau} w_{t+\tau} \right)$ and $q_t H^* = \frac{(1-\mu)(1-\eta)}{1+\beta} \left(n_t w_t + \sum_{s=1}^\infty \prod_{\tau=1}^s \frac{1}{1+r_{t+\tau}} n_{t+\tau} w_{t+\tau} \right)$, that is, aggregate expenditure on real money balances (real properties) is $\mu(1-\eta)/(1+\beta)$ ($(1-\mu)(1-\eta)/(1+\beta)$) of the discounted present value of the aggregate labor income from now on, which completes the proof.

Proof of Lemma 3

Dividing (30) by $n_t k_{t+1}$,

$$\beta x_t - (1+\beta-\eta)\psi_{t+1}\frac{1}{1-\alpha}\frac{1}{(n_t k_{t+1})^{1-\alpha} n_{t+1}^{\alpha}} = 1-\eta \tag{33}$$

(29) for the next period implies

$$\frac{1}{(n_t k_{t+1})^{1-\alpha} n_{t+1}^{\alpha}} = \frac{\left(1-\frac{\eta}{1+\beta}\right)\alpha}{n_{t+1}k_{t+2}+\psi_{t+1}} \tag{34}$$

Substituting (34) into (33) and rearranging considering $x_{t+1} \equiv \frac{\psi_{t+1}}{n_{t+1}k_{t+2}}$,

$$x_t = \frac{1-\eta}{\beta} + \frac{1+\beta}{\beta}\frac{\alpha}{1-\alpha}\left(1-\frac{\eta}{1+\beta}\right)^2 \frac{x_{t+1}}{1+x_{t+1}} \tag{35}$$

The difference equation (35) unique positive and steady state x^* such that $\frac{1-\eta}{\beta} < x^* < \frac{1-\eta}{\beta} + \frac{1+\beta}{\beta}\frac{\alpha}{1-\alpha}\left(1-\frac{\eta}{1+\beta}\right)^2$. This steady state is unstable, hence the only x_t that is consistent with saddle point path is x^*.

Appendix F Paths of k and ψ on Table 5

Dividing (29) for each period by $n_t k_{t+1}$ and considering Lemma 2,

$$\begin{aligned}
\text{up to period 0} \quad & 1 = \left(1-\frac{\eta}{1+\beta}\right)\alpha\left(\frac{1}{\bar{k}}\right)^{\alpha} - x^* \\
\text{period 1} \quad & 1 = \left(1-\frac{\eta}{1+\beta}\right)\alpha\left(\frac{(n+\gamma)\bar{k}}{(n+\Delta)k_2}\right)^{1-\alpha}\left(\frac{1}{k_2}\right)^{\alpha} - x^* \\
\text{period 2} \quad & 1 = \left(1-\frac{\eta}{1+\beta}\right)\alpha\left(\frac{(n+\Delta)k_2}{nk_3}\right)^{1-\alpha}\left(\frac{1}{k_3}\right)^{\alpha} - x^* \\
\text{period 3 and thereafter} \quad & 1 = \left(1-\frac{\eta}{1+\beta}\right)\alpha\left(\frac{k_t}{k_{t+1}}\right)^{1-\alpha}\left(\frac{1}{k_t}\right)^{\alpha} - x^* \quad \forall t \geq 3
\end{aligned}$$

These equations imply

$$\left(\frac{1}{\overline{k}}\right)^{\alpha} = \left(\frac{(n+\gamma)\overline{k}}{(n+\Delta)k_2}\right)^{1-\alpha}\left(\frac{1}{k_2}\right)^{\alpha} = \left(\frac{(n+\Delta)k_2}{nk_3}\right)^{1-\alpha}\left(\frac{1}{k_3}\right)^{\alpha} = \left(\frac{k_3}{k_4}\right)^{1-\alpha}\left(\frac{1}{k_3}\right)^{\alpha} = \cdots$$

Multiplying by $\overline{k}^{\alpha}$,

$$1 = \left(\frac{n+\gamma}{n+\Delta}\right)^{1-\alpha}\left(\frac{\overline{k}}{k_2}\right) = \left(\frac{n+\Delta}{n}\right)^{1-\alpha}\left(\frac{k_2}{\overline{k}}\right)^{1-\alpha}\left(\frac{\overline{k}}{k_3}\right) = \left(\frac{k_3}{\overline{k}}\right)^{1-\alpha}\left(\frac{\overline{k}}{k_4}\right) = \cdots \tag{F1}$$

or equivalently,

$$1 = \left(\frac{\overline{\psi}}{\psi_1}\right)^{1-\alpha}\left(\frac{\overline{k}}{k_2}\right)^{\alpha} = \left(\frac{\psi_1}{\psi_2}\right)^{1-\alpha}\left(\frac{\overline{k}}{k_3}\right)^{\alpha} = \left(\frac{\psi_2}{\psi_3}\right)^{1-\alpha}\left(\frac{\overline{k}}{k_4}\right)^{\alpha} = \cdots \tag{F2}$$

(F1) implies

$$\begin{aligned}
\frac{k_2}{\overline{k}} &= \left(\frac{n+\gamma}{n+\Delta}\right)^{1-\alpha} < 1 \\
\frac{k_3}{\overline{k}} &= \left(\frac{n+\Delta}{n}\right)^{1-\alpha}\left(\frac{k_2}{\overline{k}}\right)^{1-\alpha} = \left(\frac{n+\Delta}{n}\right)^{\alpha}\left(\frac{n+\gamma}{n+\Delta}\right)^{(1-\alpha)^2} \\
\frac{k_{t+1}}{\overline{k}} &= \left(\frac{k_t}{\overline{k}}\right)^{1-\alpha} \qquad \forall t \geq 3
\end{aligned}$$

and (F2) implies

$$\begin{aligned}
\frac{\psi_1}{\overline{\psi}} &= \left(\frac{\overline{k}}{k_2}\right)^{\frac{\alpha}{1-\alpha}} = \left(\frac{n+\Delta}{n+\gamma}\right)^{\alpha} > 1, \\
\frac{\psi_2}{\psi_1} &= \left(\frac{\overline{k}}{k_3}\right)^{\frac{\alpha}{1-\alpha}} = \left(\frac{n}{n+\gamma}\right)^{\alpha}\left(\frac{n+\gamma}{n+\Delta}\right)^{\alpha} < 1 \\
\frac{\psi_t}{\psi_{t-1}} &= \left(\frac{\overline{k}}{k_{t+1}}\right) = \left(\frac{\overline{k}}{k_t}\right)^{\alpha} = \left(\frac{\psi_{t-1}}{\psi_{t-2}}\right)^{1-\alpha} \qquad \forall t \geq 3
\end{aligned}$$

The last equation implies that ψ_t converges to the limit, $\frac{n}{n+\gamma}\overline{\psi}$, that is, $\lim_{t\to\infty}\frac{\psi_t}{\overline{\psi}} = \lim_{t\to\infty}\frac{\psi_t}{\psi_1}\frac{\psi_1}{\overline{\psi}} = \left(\frac{\psi_2}{\psi_1}\right)^{1/\alpha}\frac{\psi_1}{\overline{\psi}} = \frac{n}{n+\gamma}$.